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A NEW METHOD OF CALCULATING ROOTS OF ALGEBRAIC EQUATIONS

By G. S. SMITH

A method of calculating real roots by a simple division process when an integral approximation is known is described. It is very powerful and economical in labour as well as being of extreme simplicity. It surpasses Horner's method in these respects and in many applications it is preferable to the Newton-Raphson method and others. A root is expressed directly in any scale of notation or, alternatively, as a continued fraction, and the processes involved would appear to be suitable for use in an automatic computer.

When an integral approximation a to a root of the equation $f(x) = 0$, where $f(x)$ is a polynomial of any degree, has been found, a simple transformation, $x = a - yf(a)$, where $yf(a)$ is less than unity, will give the equation $1 = yF(y)$, where $F(y)$ is a function of y.

We will first suppose that a and y are positive quantities and that $f(a)$ is negative, but the principle of the method to be described remains the same under other conditions, as will be illustrated later. The first decimal place of the root is clearly the greatest integer in $-10yf(a)$, which can be written as $\dfrac{10yf(a)}{1}$, or since $1 = yF(y)$, as $\dfrac{-10f(a)}{F(y)}$. This fraction will, in general, show purely numerical terms in both numerator and denominator. If we replace these by multiples of unity, i.e. of $yF(y)$, then cancel y, and repeat as necessary the process of substituting $yF(y)$ for 1 in the purely numerical terms and cancelling y, we shall necessarily obtain a fraction that will yield an integer and a residual fraction in which the

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When an integral approximation a to a root of the equation $f(x) = 0$, where $f(x)$ is a polynomial of any degree, has been found, a simple transformation, $x = a - yf(a)$, where $yf(a)$ is less than unity, will give the equation $1 = yF(y)$, where $F(y)$ is a function of y.

We will first suppose that a and y are positive quantities and that $f(a)$ is negative, but the principle of the method to be described remains the same under other conditions, as will be illustrated later. The first decimal place of the root is clearly the greatest integer in $-10yf(a)$, which can be written as $\dfrac{-10yf(a)}{1}$, or since $1 = yF(y)$, as $\dfrac{-10f(a)}{F(y)}$. This fraction will, in general, show purely numerical terms in both numerator and denominator. If we replace these by multiples of unity, i.e. of $yF(y)$, then cancel y, and repeat as necessary the process of substituting $yF(y)$ for 1 in the purely numerical terms and cancelling y, we shall necessarily obtain a fraction that will yield an integer and a residual fraction in which the

numerator is recognizably less than the denominator. The proof of this assertion depends on the properties of recurring series. The integer obtained is the figure corresponding to the first decimal place of the root. The next decimal place is found by carrying out a similar process on the residual fraction after multiplying it by ten, and successive decimal places are obtained in the same way.

Thus, with $x^3 + 3x - 5 = 0$ we have $1 = y(6 + 3y + y^2)$ with $x = 1 + y$, where y is less than unity, so that we have to find first the integral part of $10y$.

$$\frac{10y}{1} = \frac{10}{6 + 3y + y^2} = \frac{60 + 30y + 10y^2}{39 + 19y + 6y^2} = 1 + \frac{21 + 11y + 4y^2}{39 + 19y + 6y^2}$$

For the second decimal place we have

$$\frac{210 + 110y + 40y^2}{39 + 19y + 6y^2} = \frac{1370 + 670y + 210y^2}{253 + 123y + 39y^2}$$

$$= 5 + \frac{105 + 55y + 15y^2}{253 + 123y + 39y^2}$$

and the root can be developed figure by figure as 1·15417149 . . . .

At any stage in the process the approximate value already found for y can be used in the residual fraction to give several more figures of the answer. Thus in the above after the first stage in which the value 0·1 has been found the insertion of the value 0·1 for y in the residual fraction will yield the next three figures 541 of the root by simple division, and if the value 0·15 obtained after the second stage is inserted in the succeeding fraction the next four figures 4171 are obtained.

Next, suppose that a and y and also $f(a)$ are all positive. The root is then less than a and the decimal obtained by means of the process described above has to be subtracted from a to give the desired root.

Thus, with $x^3 - 9x^2 + 23x - 14 = 0$, we have for the root near, but somewhat less than, 1, the derived equation $1 = y(8 + 6y + y^2)$ with $x = 1 - y$ so that

$$\frac{10y}{1} = \frac{10}{8 + 6y + y^2} = \frac{80 + 60y + 10y^2}{70 + 49y + 8y^2} = 1 + \frac{10 + 11y + 2y^2}{70 + 49y + 8y^2}$$

and

$$\frac{100 + 110y + 20y^2}{70 + 49y + 8y^2} = \frac{910 + 620y + 100y^2}{609 + 428y + 70y^2}$$

$$= 1 + \frac{301 + 192y + 30y^2}{609 + 428y + 70y^2}$$

By continuing the process we have for the value of y 0·114907... so that the desired root is $1 - 0{\cdot}114907\ldots$ or 0·885093... .

To illustrate a case where a and $f(a)$ are positive and y is negative we take the equation $x^3 - 9x^2 + 23x - 14 = 0$ for the purpose of finding the value of the root that is near but somewhat greater than 3. We have $x = 3 - y$ and $1 = y(-4 + y^2)$, and we require first the greatest integer in $\dfrac{-10y}{1}$.

$$\frac{-10y}{1} = \frac{-10}{-4 + y^2} = \frac{40 - 10y^2}{16 + y - 4y^2} = \frac{-160 - 10y + 40y^2}{-63 - 4y + 16y^2}$$

$$= 2 + \frac{-34 - 2y + 8y^2}{-63 - 4y + 16y^2}$$

Then

$$\frac{-340 - 20y + 80y^2}{-63 - 4y + 16y^2} = 5 + \frac{-25}{-63 - 4y + 16y^2}$$

and so on, giving the root as 3·2541015... .

When a is positive and y is negative occasions may arise when the apparent value of the integer is incorrect. The correct value will, however, be obtained if the process of substituting for unity is continued. Even if this is not done the value will be automatically corrected at the next stage. Thus with $2x^2 - 14x + 23 = 0$ and taking a as 2 we have $x = 2 - 3y$ and $1 = y(-6 - 6y)$. Now $\dfrac{-30y}{1} = \dfrac{180 + 180y}{30 + 36y}$ would yield $5 + \dfrac{30}{30 + 36y}$ but 5 is not the greatest integer in $\dfrac{180 + 180y}{30 + 36y}$. The next stage gives

$$\frac{300}{30 + 36y} = \frac{50}{5 + 6y} = \frac{-300 - 300y}{-24 - 30y} = \frac{1500 + 1800y}{114 + 144y}$$

$$= \frac{-7200 - 9000y}{-540 - 684y} = 13 + \frac{-180 - 108y}{-540 - 684y}$$

so that the beginning of the root is actually 2·63... .

When there are two real roots that are not widely separated the method described will give one of these roots but the number of operations required may be great. This defect is common to all methods. The original equation should be transformed into another the roots of which are the squares or some known multiple of those of the original equation.

The method is not restricted to numbers in the denary scale. The root can be expressed directly in any scale of notation. Thus, suppose we wish to express $\sqrt{7}$ in the binary scale. We have $x^2 - 7 = 0$ which with $a = 2$ requires the substitution $x = 2 + 3y$, giving $1 = y(4 + 3y)$. The first binary place after the 'decimal'

point is the greatest integer in $2 \times 3y$. Hence we have

$$\frac{6y}{1} = \frac{6}{4+3y} = \frac{24+18y}{19+12y} = 1 + \frac{5+6y}{19+12y}$$

$$2 \times \frac{5+6y}{19+12y} = 0 + \frac{10+12y}{19+12y}$$

$$2 \times \frac{10+12y}{19+12y} = 1 + \frac{1+12y}{19+12y}$$

$$2 \times \frac{1+12y}{19+12y} = \frac{32+6y}{88+57y} = 0 + \frac{32+6y}{88+57y}$$

$$2 \times \frac{32+6y}{88+57y} = 0 + \frac{64+12y}{88+57y}$$

$$2 \times \frac{64+12y}{88+57y} = \frac{536+384y}{409+264y} = 1 + \frac{127+120y}{409+264y}$$

$$2 \times \frac{127+120y}{409+264y} = 0 + \frac{254+240y}{409+264y}$$

$$2 \times \frac{254+240y}{409+264y} = 1 + \frac{99+216y}{409+264y}$$

so that the root is approximately 10·10100101... . The approximation can be improved considerably by substituting the approximate value of y, i.e. one third of 0·10100101 in the last fraction to give the succeeding figures 010011111110. The value of $\sqrt{7}$ is thus found to be 10·10100101010011111110... which is in error only in the last place. By means of this substitution the number of known figures in the result has been doubled.

The method can be used also to express a root directly as a continued fraction, and for this the calculations are even simpler.

Thus, to express $\sqrt{13}$ as a continued fraction we have first $x^2 - 13 = 0$ which with $x = 3 + 4y$ gives $1 = y(6 + 4y)$. We have then

$$x = 3 + \frac{1}{\dfrac{1}{4y}}$$

$$\frac{1}{4y} = \frac{6+4y}{4} = \frac{40+24y}{24+16y} = \frac{5+3y}{3+2y} = 1 + \frac{2+y}{3+2y}$$

$$\frac{3+2y}{2+y} = 1 + \frac{1+y}{2+y}$$

$$\frac{2+y}{1+y} = 1 + \frac{1}{1+y}$$

$$1 + y = 1 + \frac{y}{1}$$

$$\frac{1}{y} = \frac{6 + 4y}{1} \qquad = 6 + 4y$$

after which the cycle recommences. Hence

$$\sqrt{13} = 3 + \underset{\times}{\frac{1}{1+}}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \underset{\times}{\frac{1}{6+}}$$

Again, to express the root near 2 of $x^3 - 2x - 5 = 0$ as a continued fraction we have $x = 2 + y$ and $1 = y(10 + 6y + y^2)$. Then

$$\frac{1}{y} = \frac{10 + 6y + y^2}{1} = \frac{106 - 61y + 10y^2}{10 + 6y + y^2} = 10 + \frac{6 + y}{10 + 6y + y^2}$$

$$\frac{10 + 6y + y^2}{6 + y} = \frac{106 + 61y + 10y^2}{61 + 36y + 6y^2} = 1 + \frac{45 + 25y + 4y^2}{61 + 36y + 6y^2}$$

$$\frac{61 + 36y + 6y^2}{45 + 25y + 4y^2} = 1 + \frac{16 + 11y + 2y^2}{45 + 25y + 4y^2}$$

$$\frac{45 + 25y + 4y^2}{16 + 11y + 2y^2} = 2 + \frac{13 + 3y}{16 + 11y + 2y^2}$$

$$\frac{16 + 11y + 2y^2}{13 + 3y} = \frac{171 + 98y + 16y^2}{133 + 78y + 13y^2} = 1 + \frac{38 + 20y + 3y^2}{133 + 78y + 13y^2}$$

and the root is developed as

$$2 + \frac{1}{10+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{12+}\ \cdots$$

In any of the applications of the method the operations can be shortened by the use of coefficients in a manner similar to that employed in the Horner scheme.

G. S. Smith

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A PROBLEM IN PROJECTIVE GEOMETRY

T. G. Room

The theorem to be discussed in this note depends only on the plane incidence axioms (with Desargues), the Pappus axiom, and the property that a point and its harmonic conjugate are in general distinct. When algebra is applied to the geometry it may be taken over any commutative field of characteristic other than 2.

I use the notations:

"$K \in AB$" for "K lies on AB"

"$B = AL \cap VQ$" for "B is the meet of AL and VQ", and

"$Q = (L, L')/P$" for "Q is the harmonic conjugate of P with regard to L and L'."

In writing of an involution on a line I mean the system of pairs of points determined by the quadrangle construction from two given pairs.

The theorem for which a proof is sought is this:

THEOREM I. GIVEN: *on a line u three distinct pairs of points L, L'; M, M'; N, N' of an involution $\mathfrak{s}$ and an arbitrary point P on u (not a double-point of $\mathfrak{s}$),*

CONSTRUCT:

$Q = (L, L')/P$,

$R = (M, M')/Q$,

$S = (N, N')/R$, *and*

K, K' *that pair of* $\mathfrak{s}$ *for which* $K' = (P, S)/K$

THEOREM:

(i) *the pair K, K' is independent of the choice of P, and*

(ii) *the points K, K' are rationally constructable.*

By "rationally constructable" I mean that the points can be constructed from the given points by drawing a finite number of lines through pairs of points and naming the intersections of pairs of them, in the same fashion as N' can be constructed from L, L'; M, M'; N by the quadrangle construction. It is to be remarked that as defined in the construction above, K, K' form the common pair of two involutions, and so might be expected to be determined only as solutions of a quadratic equation. It is to be shown eventually that the equation does in fact have roots rationally determined from the parameters of L, L'; M, M'; N.

In Theorem III a construction is described that determines K, K', but I have not been able to find a projective geometric proof that the figure has the required properties.

The problem arose from an attempt to find a projective proof of the fundamental relation among "reflection" (or "symmetry") transformations in the plane, where the reflection transformation with regard to (or "in") a line l, say $\mathbf{R}_l$, is defined thus:

"$B = \mathbf{R}_l A$" : "B is the reflection of A in l", i.e., B is such that l is the right-bisector of the segment AB.

The fundamental relation among reflections is this:

THEOREM II. GIVEN *l, m, n three lines concurrent in V, and A an arbitrary point (not V),*

CONSTRUCT

$B = \mathbf{R}_l A$,

$C = \mathbf{R}_m B$,

$D = \mathbf{R}_n C$, and

k such that $D = \mathbf{R}_k A$ *(i.e., k is the right-bisector of the segment AD)*

THEOREM

(i) *k passes through V*

(ii) *k is independent of the choice of A.*

Part (i) of the theorem is a consequence of the theorem on the concurrence of the right-bisectors of the sides of a triangle, but part (ii) requires in its proof congruence properties (equal angles), and these cannot be expressed projectively in terms of the ideal line and the involution of perpendicular directions alone. The significance of part (ii) is that the three concurrent lines l, m, n determine a unique fourth line k concurrent with them, which is such that

$$\mathbf{R}_k A = \mathbf{R}_n \mathbf{R}_m \mathbf{R}_l A$$

for all A, i.e., in terms of the operators $\mathbf{R}$,

$$\mathbf{R}_k = \mathbf{R}_n \mathbf{R}_m \mathbf{R}_l.$$

Since $\mathbf{R}^2 = 1$, we may write this as

$$\mathbf{R}_n \mathbf{R}_k = \mathbf{R}_m \mathbf{R}_l.$$

In Euclidean language the resultant of $\mathbf{R}_l$ and $\mathbf{R}_m$ is a "rotation" round V through twice the angle between l and m. The same rotation is the resultant of $\mathbf{R}_k$ and $\mathbf{R}_n$ provided k and n also pass through V and the angle between them is the same as that between l and m.

Take u to be the ideal line and $\mathbf{J}$ the involution on it defining perpendicular directions. The reflection transformation $\mathbf{R}_l$ is then defined thus:

"If $l \cap u = L$,

and $\mathbf{J}L = L'$,

then $B = \mathbf{R}_l A$,

is defined by: $B \in AL'$ and $B = (L', l \cap AL')A$". The projective form of Theorem II is then

THEOREM III. GIVEN

(i) *u and on it an involution* $\mathbf{J}$,

(ii) *l, m, n concurrent in V, $(V \notin u)$,*

$$(l, m, n) \cap u = L, M, N$$

$$\mathbf{J}(L, M, N) = L', M', N'$$

(iii) *an arbitrary point $A(A \neq V)$,*

CONSTRUCT

$$B : B \in AL',\ B = (L', AL' \cap l)/A$$

$$C : C \in BM',\ C = (M', BM' \cap m)/B$$

$$D : D \in CN',\ D = (N', CN' \cap n)/C$$

THEOREM: *there is a unique line k such that*

(i) *k passes through V, and if $k \cap u = K$, $AD \cap u = K'$, then*

(ii) $\mathbf{J}K = K'$

(iii) $K' = (A, D)/K$.

Theorem I is obtained from Theorem III by projecting from V on to u, so that a proof of Theorem III provides an immediate proof of Theorem I. It is not difficult to prove conversely Theorem III from Theorem I. Theorem III provides the finite projective construction of the pair K, K' from the three pairs L, L'; M, M'; N, N' of J.

To prove Theorem I algebraically take the parameters of points $L, \ldots, P, \ldots$, etc. on u to be $l, \ldots, p, \ldots$, etc. over any commutative field of characteristic $\neq 2$. It can be proved from the quadrangle construction that x, x' are a pair in the involution determined by l, l'; m, m' if

$$\begin{vmatrix} xx' & x + x' & 1 \\ ll' & l + l' & 1 \\ mm' & m + m' & 1 \end{vmatrix} = 0.$$

By suitable changes of parameters and renaming of coefficients this relation can be successively reduced to

$$xx' + \kappa(x + x') + \kappa' = 0$$

$$\overline{xx'} + \kappa' - \kappa^2 = 0$$

$$\overline{\overline{xx'}} = h'/\rho^2 = h.$$

Beyond this we cannot go without further knowledge of the field. If the field contains no non-squares, then h may be taken to be 1;

if it is the real field we may take either $h = 1$ or $h = -1$ (this being the Euclidean case). To preserve the generality of the relation I keep h unspecified.

After the reduction has been effected, if $l, l', \ldots$ are the new parameters, we have

$$ll' = mm' = nn' = h.$$

If we write

$$\lambda = \tfrac{1}{2}(l + l'),\ \mu = \tfrac{1}{2}(m + m'),\ \nu = \tfrac{1}{2}(n + n'),$$

the harmonic relations of Theorem I give:

$$q = (\lambda p - h)/(p - \lambda)$$

$$r = (\mu q - h)/(q - \mu) = (\alpha p + \beta h)/(\beta p + \alpha),$$

$$\alpha = \lambda\mu - h,\ \beta = \lambda - \mu,$$

$$s = (\nu r - h)/(r - \nu) \qquad = (\gamma p - \delta h)/(\delta p - \gamma),$$

$$\gamma = h(\lambda - \mu + \nu) - \lambda\mu\nu,\ \delta = h - (\mu\nu - \nu\lambda + \lambda\mu),$$

and k, k' have to satisfy the two conditions

$$kk' = h$$

$$\tfrac{1}{2}(k + k') = (ps + h)/(p + s)$$

$$= (p^2\gamma - h\gamma)/(p^2\delta - h\delta) = \gamma/\delta.$$

Thus, certainly, k, k' are independent of p, but it is not evident that they are rationally determined, since, so far, they are the roots of a quadratic equation with apparently unrestricted coefficients. However, when we replace λ by $\tfrac{1}{2}(l + l') = \tfrac{1}{2}(l^2 + h)/l$, we find that in fact the quadratic is rationally soluble, the roots being

$$k.k' = \frac{h(l - m + n) - lmn}{h - (mn - nl + lm)},\ \frac{h^2 - h(mn - nl + lm)}{h(l - m + n) - lmn}.$$

There is another curious facet to this theorem: in the Euclidean case we know that the angular regions determined by the lines l, m are "congruent" to those determined by n, k. If we define two geometrical figures to be congruent when there is a sequence of reflection transformations which transforms one into the other, then in the projective form we cannot prove that the regions are congruent. In the Euclidean case the transformations relating one set to the other can be reduced to the reflection in one or other of the two lines which bisect the angles between l and n. In projective language these transformations, if they exist, are determined by the pair of points of the involution **J** which harmonically separates L and N. The points of such a pair would be determined by

$$xx' = h$$

$$\tfrac{1}{2}(x + x') = (ln + h)/(l + n),$$

and consequently, in any given field, they need not be rationally determinable, and, if the field contains non-squares, they need not even exist.

T. G. R.

University of Sydney

"N. BOURBAKI"

J. A. P. Hall

In his Presidential Address to the Mathematical Association in January 1957 (Mathematical Gazette *41*, 1957, 162–8) Professor G. Temple, F.R.S. said:—

> "It is well known how the development of analysis in this country has been decisively influenced by G. H. Hardy's epoch-making textbook entitled *A Course of Pure Mathematics*. And it seems probable that our whole approach to algebra, analysis and topology will be transformed by the astonishing series of monographs by that group of French mathematicians who write under the collective synonym [sic] of "Bourbaki"."

It is the object of this Note to give some information and references concerning Bourbaki.

There is an informative and amusing article by Halmos in the Scientific American for May 1957. The Bourbaki group is not of fixed composition and has an age limit of 50. The founders included André Weil and Jean Dieudonné, and to date, all members are, or have been, French, except for S. Eilenberg.

The point of view of the group may be read (in English) in a brilliant article by Bourbaki "himself," entitled The Architecture of Mathematics, in the American Mathematical Monthly, *57*, 1950, pp. 221–232. Briefly, they seek a unitary view of mathematics by analysing the procedures of mathematics, instead of starting from a priori philosophical views concerning the relations of mathematics with the external world and the world of thought. They believe that "something like a central nucleus that is more coherent than it has ever been" now exists. "Deductive reasoning" is NOT a unifying principle but is only the vehicle or language of mathematical thought. By means of the axiomatic method they seek to reveal the common ideas of mathematical theories through *the notion of structure*.

A mathematical structure consists of (i) sets of elements and sets of sets, (ii) one or more relations into which these elements or sets enter, and (iii) axioms or explicitly stated conditions satisfied by the relation or relations. The logical consequences of these are

deduced without any hypothesis about the nature of the elements.

"*The great types of structures*" or "*mother-structures*" are

(a) *Algebraic Structures* in which the relations are "laws of composition" (i.e. relations between three elements determining the third uniquely in terms of the other two);

(b) *Order Structures*, in which the relations are order relations; as, for example, the real number structure, in which the order relation is $<$; and

(c) *Topological Structures*, which "furnish an abstract mathematical formulation of the intuitive concepts of neighbourhood, limit and continuity, to which we are led by our idea of space."

Each structure has its own "arsenal of general theorems"; these are available once the underlying type of structure is recognised in any situation. For example, when it was discovered that the set of complex numbers had the topological structure of the Euclidean plane, the theorems of plane Euclidean geometry became available for the study of complex numbers; and analysis leapt forward.

Each mother-structure contains a diversity of types of structure obtained by adjoining supplementary axioms.

There are also *multiple structures* which arise when "two or more of the great mother-structures" are "combined organically by one or more axioms which set up a connection between them." (For example, the combination of order structures and algebraic structures leads to the theory of integration.)

When the elements of the sets under consideration are given certain definite individualities, the theories merge with those of classical mathematics.

The structures are not immutable; for they are not finished edifices; and new ones may emerge. Also, there are many isolated results not yet connected satisfactorily with known structures.

Bourbaki concludes the article with the striking remark that the "internal life of mathematics ... is like a big city, whose outlying districts and suburbs encroach incessantly, and in a somewhat chaotic manner, on the surrounding country, while the centre is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their labyrinths of alleys, and projecting towards the periphery new avenues, more direct, broader and more commodious".

The unification of mathematics is being carried out by Bourbaki through the medium of "the astonishing series of monographs" referred to by Professor Temple (above). So far twenty-five of

these have been published; and the work is continuing. The overall title is ELEMENTS DE MATHEMATIQUE. All 25 volumes so far issued cover, *but do not conclude*, the *First Part: The Fundamental Structures of Analysis*. It is highly instructive to study the titles of the books and chapters so far published. (The order in which they have been published is different and is indicated by the monograph number.) Already some monographs have reached a second edition.

As Halmos says, the presentation of each subject is systematic and thorough and often includes a brilliant historical review. The fascicules of results, one of which it is intended shall be annexed to each book, are summaries containing, as far as possible, what is essential to the study of subsequent parts. It is important to note that the terminology and symbolism used are often different from those commonly adopted.

The volume of the American Mathematical Monthly referred to above (*57*, 1950) also contains (on pp. 295-306) a remarkable article on The Future of Mathematics by André Weil, one of the founders of Bourbaki. After noting the robust vitality of some of the principal branches of mathematics, he asks:

"Does this mean that mathematics is becoming a science for erudites, and that it will no longer be possible to do creative work in mathematics until one has grown gray in the harness, and exhausted from burning the midnight oil for many years in the company of dusty tomes?"

Weil answers: No. "In the future, as in the past, the great ideas must be simplifying ideas, the creator must always be one who clarifies, for himself and for others, the most complicated tissues of formulae and concepts". And he recalls Hilbert's remark "that every important advance in mathematics is related to the simplification of methods, to the disappearance of old procedures which have lost their usefulness, and to the unification of branches which were until then foreign to each other".

Since Bourbaki does not regard "deductive reasoning" as a unifying principle but as the language of mathematical thought, it is important to know that he considers logic as "no more and no less than the grammar of the language we use, a language which had to exist before the grammar could be constructed". (Proofs were constructed before the logical analysis of the structure of a proof.) Logical and mathematical reasoning are regarded as "the same". These ideas are set out in a paper on the Foundations of Mathematics for the Working Mathematician which he published in the Journal of Symbolic Logic *14*, March 1949, pp. 1–8, and in which he set out in "its pure form" "the vocabulary and grammar of the sign-language" he wished to use. This paper concluded with the words:

"On these foundations, I state that I can build up the whole of the mathematics of the present day; and, if there is anything original in my procedure it lies solely in the fact that, instead of being content with such a statement, I proceed to prove it in the same way as Diogenes proved the existence of motion; and my proof will become more and more complete as my treatise grows."

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JOHN SMITH'S PROBLEM

By T. W. CHAUNDY and J. E. BULLARD

In the autumn of 1693 the Writing Master of Christ's Hospital, one John Smith, asked Samuel Pepys the diarist, who was a Governor of Christ's Hospital and eventually became its Vice-President, for the answer to the following question:

The Question.

A—has 6 dice in a box, with which he is to fling a 6.

B—has in another box 12 dice, with which he is to fling 2 sixes.

C—has in another box 18 dice, with which he is to fling 3 sixes.

Q—Whether B and C have not as easy a taske as A at even luck?

In this question "to fling a 6" has to be interpreted as meaning "to fling at least one 6"; and similarly "to fling 2/3 sixes" must be interpreted as meaning "to fling at least 2/3 sixes".

We know from Pepys that a project by "Mr. Neale the Groom-Porter" for a lottery-loan to be secured by a new tax on salt (a suggestion which was in fact adopted in the following year) had at this time "extinguished every other talk but what relates to the doctrine of determining between the true proportion of the hazards incident to this or that given chance or lot": and it is probable that Smith's interest in such questions was not just academic, especially as the task of drawing tickets for the major lotteries was commonly given to boys from the School. That Smith took opportunities of making money on the side is shown by his subsequent dismissal from the Hospital's employment because he was found to be charging boys for the pens and paper used in the Writing School; Pepys was then foremost in demanding his dismissal, but previously the writing master seems to have stood high in his good graces, and Pepys, in his anxiety to help him with his problem, sent him to

Cambridge with a letter of introduction to the Lucasian Professor as the person most likely to provide the solution. Pepys also put the problem to a civil service colleague, George Tollet, Secretary to the Lords Commissioners for taking the Public Account, of whose mathematical abilities he obviously had a high opinion since at a critical moment in the history of the Hospital's Mathematical Foundation he had him appointed as external examiner to report on the boys' mathematical attainments.

The Lucasian Professor at this time was Isaac Newton. Newton had already interested himself in the affairs of the Hospital (of which he later became a Governor), both by advising the Governors on the appointment of Masters for the Mathematical Foundation, and by drawing up a revised syllabus of instruction. His long but very business-like letter about the syllabus can be found printed as an Appendix to Edleston's "Correspondence of Sir Isaac Newton and Professor Cotes etc." (London, 1850).

Pepys' letter to Newton was dated November 22nd. Four days later Newton sent an interim reply asking about the interpretation to be attached to the question, and stating that on his interpretation (which seems to be the only natural one) the six-throw man had a better chance than the twelve-throw man, who again had a better chance than the eighteen-throw man. When a further letter from Pepys had reassured him on the point of interpretation, he sent on 16 December a letter confirming his result and giving details of his mode of calculation. He says in this letter that the chance of a man gaining at least one success in six throws is 31,031 : 46,656; and that the chance of his gaining at least two successes in twelve throws is 1,346,704,211 : 2,176,782,336. He does not give figures for the eighteen-throw case, but his method of calculation gives the result 60,666,401,980,916 : 101,559,956.668,416. The same results (including the result for eighteen throws) were obtained by Tollet. Pepys was puzzled by Newton's conclusions (though he accepted them), and sent further questions to Newton on the 21st December, getting a reply on the 23rd.

The guard-book copies of Pepys' letters to Newton and Tollet, together with the original replies, passed into the Pepys-Cockerell collection. This was sold at Sothebys in 1931, and the correspondence is now in private ownership in the U.S.A. Before the sale the letters in the collection had been printed in two volumes under the editorship of J. R. Tanner (G. Bell and Sons Ltd., 1926), and the correspondence about John Smith's problem will be found on pp. 72/90 of the first volume. The first two of Newton's letters have since been reprinted on pp. 45/8 of "The Gentle Art of Mathematics" by D. Pedoe (reviewed in the issue of the Gazette for October, 1959): and the correspondence will appear again in the Third Volume of the complete edition of letters to and from Sir

Isaac Newton now being edited on behalf of the Royal Society by Professor Turnbull F.R.S.

Newton's explanation of his calculations involves writing down eight series or progressions

(1) series of integers from 1 onwards;

(2) series for $\binom{n}{2}$ obtained by adding entries in the first series;

(3) series of nth powers of 6;

(4) series of nth powers of 5;

(5) series of $(n-1)$th powers of 5;

(6) series for $n \cdot 5^{n-1}$ obtained by multiplying the entries in series (1) and (5);

(7) series of $(n-2)$th powers of 5;

(8) series for $\binom{n}{2} \cdot 5^{n-2}$ obtained by multiplying the entries in series (2) and (7).

Newton explains that out of the total number of chances given for n dice by the entry under n in row (3), the entry in row (4) gives the number of chances without a six; the entry in row (6) gives the number with only one six: and the entry in row (8) gives the number with just two sixes. The rest is easy. Such an investigation was well within the competence of any professional mathematician (or, as Tollet's success shows, of any capable amateur) who had read his Pascal: but the way in which Newton systematises the calculation so as to enable any such problem to be solved by the one general method is very characteristic.

Newton's calculations suggest that, more generally, the chance of getting at least n sixes in $6n$ throws is likely to decrease steadily as n increases, whatever n, a possibility put to us by Professor Turnbull. Here we widen the problem by considering a die with s faces all equally likely where, of course $s \geqslant 2$ since otherwise we should be betting on certainties, and we write $f(t, n)$, $g(t, n)$ for the respective chances that, in t throws, a selected face does or does not turn up at least n times, so that $f + g = 1$. It seems a little simpler to work with the chance of failure g, and we proceed to prove that

(i) $g(sn, n)$ increases with n for fixed s;

(ii) $g(sn, n)$ increases with s for fixed n;

(iii) $g(sn, n) < \frac{1}{2}$;

(iv) $g(2n, n) \to \frac{1}{2}$ as $n \to \infty$.

Write $x = s^{-1}$, $y = 1 - s^{-1}$. Then, in the expansion of $(x + y)^t$, the sum of the terms containing x^n gives $f(t, n)$ while the sum of the remaining terms, i.e., those containing y^{t-n+1}, gives $g(t, n)$.

With any positive integers n, q write $S_n(q, x)$ for the sum of the first n terms in the binomial expansion of $(1 - x)^{-q}$, so that

$$S_n(q, x) = 1 + qx + \frac{q(q+1)}{2!}x^2 + \ldots + \frac{(q+n-2)!}{(n-1)!(q-1)!}x^{n-1}, \tag{1}$$

the terms that follow all containing x^n.

Again, in the expansion of $(x + y)^{n+q-1}$ every term contains either x^n or y^q, but no term contains both. Hence we may write

$$(x + y)^{n+q-1} = x^n U + y^q V, \tag{2}$$

where U, V are polynomials in x, y. With $x + y = 1$ we get from (2) that

$$(1 - x)^{-q} = V + x^n(1 - x)^{-q}U,$$

so that $V = S_n(q, x)$, and, since by symmetry $U = S_q(n, 1 - x)$, we may note in passing the identity

$$(1 - x)^q S_n(q, x) + x^n S_q(n, 1 - x) = 1.$$

From (2) we see further that

$$f(n + q - 1, n) = x^n U, \quad g(n + q - 1, q) = y^q V,$$

so that, when $q = (s - 1)n + 1$,

$$g(sn, n) = y^q S_n(q, x). \tag{3}$$

Changing n to $n + 1$ increases q by $s - 1$, so that to get (i) we have to show that

$$y^{s-1}S_{n+1}(q + s - 1, x) > S_n(q, x). \tag{4}$$

Now, from (1),

$$(1 - x)S_{n+1}(q + 1, x) = S_{n+1}(q, x) - \frac{(q+n)!}{n!q!}x^{n+1},$$

$$(1 - x)S_{n+1}(q + 2, x) = S_{n+1}(q + 1, x) - \frac{(q+n+1)!}{n!(q+1)!}x^{n+1},$$

and so on. Multiplying by appropriate powers of $y = 1 - x$ and adding we get

$$\begin{aligned} y^{s-1}S_{n+1}(q + s - 1, x) &= S_{n+1}(q, x) - \frac{(q+n)!}{n!q!}x^{n+1}\left\{1 + \frac{q+n+1}{q+1}y\right. \\ &\quad \left. + \ldots + \frac{(q+n+1)\ldots(q+n+s-2)}{(q+1)\ldots(q+s-2)}y^{s-2}\right\} \end{aligned}$$

But, again from (1),

$$S_{n+1}(q, x) = S_n(q, x) + \frac{(q+n-1)!}{n!(q-1)!}x^n.$$

Thus to prove (4) we have to show that

$$1 > \frac{q+n}{q} x\left\{1 + \frac{q+n+1}{q+1} y + \ldots + \frac{(q+n+1)\ldots(q+n+s-2)}{(q+1)\ldots(q+s-2)} y^{s-2}\right\}. \quad (5)$$

Now, when $r \geqslant 0$,

$$\frac{q+n+r}{q+r} < \frac{q+n-1}{q-1} = \frac{s}{s-1} = y^{-1}.$$

Thus every term in the braces $\{\ \}$ is less than or equal to unity, and there are $s-1$ of them. The expression on the right of (5) is therefore less than $(s-1)x/y$, which equals unity. Thus (5) is proved, and this completes the proof of (i).

To prove (ii) seems less easy. We now give s continuous variation, and therefore q, x, y are differentiable. In

$$g(sn, n) = (1-x)^q S_n(q, x)$$

we differentiate separately in x and q. In x we have

$$\frac{\partial}{\partial x} S_n(q, x) = q S_{n-1}(q+1, x),$$

and so

$$\begin{aligned}
&\frac{\partial}{\partial x}\left\{(1-x)^q S_n(q, x)\right\} \\
&= q(1-x)^{q-1}\{(1-x)S_{n-1}(q+1, x) - S_n(q, x)\} \\
&= q(1-x)^{q-1}\left\{S_{n-1}(q, x) - \frac{(q+n-2)!}{(n-2)!q!} x^{n-1} - S_n(q, x)\right\} \\
&= -q(1-x)^{q-1}x^{n-1}\left\{\frac{(q+n-2)!}{(n-2)!q!} + \frac{(q+n-2)!}{(n-1)!(q-1)!}\right\} \\
&= -\frac{(q+n-1)!}{(n-1)!(q-1)!} x^{n-1}(1-x)^{q-1} \qquad (6)
\end{aligned}$$

To differentiate in q we write

$$(1-x)^{-q} = \sum_0^\infty Q_r x^r,$$

so that

$$\begin{aligned}
\sum_0^\infty \frac{\partial Q_r}{\partial q} x^r &= -(1-x)^{-q}\log(1-x) \\
&= \left(\sum_0^\infty Q_r x^r\right)(x + \tfrac{1}{2}x^2 + \ldots \text{ to } \infty),
\end{aligned}$$

and hence, identifying coefficients, we have

$$\frac{\partial Q_r}{\partial q} = Q_{r-1} + \tfrac{1}{2}Q_{r-2} + \ldots + r^{-1}Q_0.$$

For remainders in $-\log(1-x)$ write

$$L_r = \frac{x^r}{r} + \frac{x^{r+1}}{r+1} + \ldots \text{ to } \infty \qquad (r = 1, 2, \ldots),$$

so that

$$L_1 + L_2 + \ldots + \text{to } \infty = x + x^2 + \ldots + \text{to } \infty = x(1-x)^{-1}. \quad (7)$$

Then

$$\begin{aligned}
&\frac{\partial}{\partial q}\{(1-x)^q S_n(q, x)\} \\
&= (1-x)^q\{S_n(q, x)\log(1-x) + \frac{\partial}{\partial q} S_n(q, x)\} \\
&= -(1-x)^q \sum_0^{n-1} \{Q_r x^r(x + \tfrac{1}{2}x^2 + \ldots + \text{to } \infty) \\
&\qquad -(Q_{r-1} + \tfrac{1}{2}Q_{r-2} + \ldots + r^{-1}Q_0)x^r\} \\
&= -(1-x)^q(x^{n-1}Q_{n-1}L_1 + x^{n-2}Q_{n-2}L_2 + \ldots + Q_0 L_n). \qquad (8)
\end{aligned}$$

Now

$$q = (s-1)n + 1, \qquad x = s^{-1},$$

and so

$$\frac{\partial g(sn, n)}{\partial s} = -x^2 \frac{\partial g}{\partial x} + n\frac{\partial g}{\partial q}$$

In (6) we can write

$$\frac{(q+n-1)!}{(n-1)!\ \ (q-1)!} = (q+n-1)Q_{n-1} = nx^{-1}Q_{n-1}.$$

Then, by (6), (8),

$$\begin{aligned}
\frac{\partial g(sn, n)}{\partial s} &= nx^{n-1}(1-x)^q\{Q_{n-1}x(1-x)^{-1} \\
&\qquad -(Q_{n-1}L_1 + Q_{n-2}x^{-1}L_2 + \ldots + Q_0 x^{-(n-1)}\ L_n)\}
\end{aligned}$$

But, since $q = (s-1)n + 1$, we have, when $r < n$,

$$\frac{Q_r}{Q_{r-1}} = \frac{q+r-1}{r} = \frac{(s-1)n + r}{r} > s = x^{-1}.$$

Thus

$$Q_{n-1} > x^{-1}Q_{n-2} > \ldots > x^{-(n-1)}Q_0.$$

Hence, by (7),

$$\begin{aligned}
Q_{n-1}x(1-x)^{-1} &> Q_{n-1}(L_1 + \ldots + L_n) \\
&> Q_{n-1}L_1 + Q_{n-2}x^{-1}L_2 + \ldots + Q_0 x^{-(n-1)}L_n
\end{aligned}$$

and so $g(sn, n)$ increases steadily with s.

To prove (iii) it is enough to show that $f(sn, n) > g(sn, n)$. Here we go back to the defining forms of f and g. Expanding $(x + y)^{sn}$ in reverse order as

$$(y + x)^{sn} = u_0 + u_1 + \ldots + u_{sn},$$

we have

$$g(sn, n) = u_0 + \ldots + u_{n-1},$$

$$f(sn, n) = u_n + \ldots + u_{sn} > u_n + \ldots + u_{2n-1}$$

since every term is positive and $sn > 2n - 1$. It is enough then to prove that

$$u_n + \ldots + u_{2n-1} > u_{n-1} + \ldots + u_0. \tag{9}$$

Now

$$\frac{u_r}{u_{r-1}} = \frac{(sn - r + 1)x}{ry} = \frac{sn - r + 1}{(s - 1)r}.$$

Thus

$$\frac{u_n}{u_{n-1}} = \frac{(s - 1)n + 1}{(s - 1)n} > 1,$$

while

$$\begin{aligned}
&\frac{u_{n+r}}{u_{n-r-1}} \Big/ \frac{u_{n+r-1}}{u_{n-r}} \\
&= \frac{u_{n+r}}{u_{n+r-1}} \cdot \frac{u_{n-r}}{u_{n-r-1}} \\
&= \frac{\{(s - 1)n - r + 1\}\{(s - 1)n + r + 1\}}{(s - 1)(n + r)(s - 1)(n - r)} \\
&> \frac{(s - 1)^2n^2 - r^2}{(s - 1)^2(n^2 - r^2)} \geq 1 \quad \text{since} \quad s \geq 2.
\end{aligned}$$

Accordingly u_{n+r}/u_{n-r-1} increases steadily with r, and we have

$$\frac{u_{2n-1}}{u_0} > \frac{u_{2n-2}}{u_1} > \ldots > \frac{u_n}{u_{n-1}} > 1.$$

Hence (9) is true since each term on the left exceeds the corresponding term on the right.

Finally in (iv) we take $s = 2$ (we could now be tossing a penny), and so $x = y = \frac{1}{2}$. Thus terms equidistant from the ends in $(y + x)^{2n}$ are equal; i.e., $u_0 = u_{2n}$, $u_1 = u_{2n-1}$, etc., but the middle term u_n belongs to f and not to g. Hence

$$f(2n, n) - g(2n, n) = u_n$$

Now

$$\begin{aligned}
u_n &= \frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n} \\
&= \frac{(2n)!}{\{2n(2n - 2) \ldots 2\}^2} \\
&= \frac{(2n - 1)(2n - 3) \ldots 1}{2n(2n - 2) \ldots 2}.
\end{aligned}$$

We may remind ourselves that this last expression tends to zero as $n \to \infty$, by writing, for any positive integer r, $(1 - \frac{1}{2}r^{-1})^{-2} = 1 + r^{-1} +$ positive terms $> 1 + r^{-1}$. Thus

$$\frac{2r-1}{2r} < \sqrt{\frac{r}{r+1}}.$$

When we take the product here from $r = 1$ to $r = n$, there is cancelling on the right, and we get

$$0 < u_n < \frac{1}{\sqrt{(n+1)}} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, as $n \to \infty$, $f(2n, n)$, $g(2n, n)$ each tend to equality at $\frac{1}{2}$. This is (iv).

It belongs to general statistical theory that, more generally for any s, $f(sn, n)$, $g(sn, n)$ still tend to equality at $\frac{1}{2}$ as $n \to \infty$. This follows at once from what has been proved above. For, from (ii), if $s > 2$,

$$g(sn, n) > g(2n, n) \quad \text{and so} \quad f(sn, n) < f(2n, n).$$

Hence also, by (iii),

$$\begin{aligned} 0 &< f(sn, n) - g(sn, n) \\ &< f(2n, n) - g(2n, n) < (n+1)^{-\frac{1}{2}}, \text{ by (iv)}, \end{aligned}$$

which gives what we want.

We add for completeness that, when $s \to \infty$, $g(sn, n)$ has the limit, given by (3) as

$$g(\infty, n) = e^{-n}\left\{1 + n + \frac{n^2}{2!} + \ldots + \frac{n^{n-1}}{(n-1)!}\right\}, \tag{10}$$

whence

$$f(\infty, n) = e^{-n}\left\{\frac{n^n}{n!} + \frac{n^{n+1}}{(n+1)!} + \ldots \text{ to } \infty\right\}. \tag{11}$$

Integration by parts gives

$$g(\infty, n) = \int_n^\infty \frac{e^{-x}x^{n-1}}{(n-1)!}\,dx, \quad f(\infty, n) = \int_0^n \frac{e^{-x}x^{n-1}}{(n-1)!}\,dx. \tag{12}$$

As above we can show that $g(\infty, n)$ increases with n and that $f(\infty, n) > g(\infty, n)$, from which we can infer that

$$g(\infty, n) \to \tfrac{1}{2} \quad \text{as} \quad n \to \infty. \tag{13}$$

But a direct proof of (13), starting from (10) or (12), has so far escaped us.

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A NOTE ON FERMAT'S THEOREM

By Louis Long

It was conjectured by Fermat that if $n > 2$, the equation

$$a^n + b^n = c^n \tag{1}$$

has no solution in positive integers. If n is composite, say $n = lm$, the equation implies $A^m + B^m = C^m$, where $A = a^l$, etc. So the insolubility of the equation with exponent m ensures the insolubility of the equation with exponent n. Since every $n > 2$ is divisible either by 4 or by a prime $p > 2$, and since the insolubility for the exponent 4 was proved by Fermat himself, it would be enough to prove the insolubility of $a^p + b^p = c^p$ for every prime $p > 2$. Most work on this equation has been limited to the so-called "first case" of Sophie Germain, in which it is assumed that none of a, b, c is divisible by p.

In the present paper we consider Fermat's equation for even exponents n. For the reasons given above, it is enough to consider the case when $n = 2p$, where p is a prime greater than 2.

We prove by simple and elementary arguments that Fermat's equation with $n = 2p$ has no solution with none of a, b, c divisible by p, if p is a prime which is not of the form $120m + 1$ or $120m + 49$.

In proving this, we can suppose that no two of a, b, c have a common factor. We write the equation as

$$b^{2p} = c^{2p} - a^{2p} = (c^2 - a^2)k, \tag{2}$$

where k is a positive integer. Put $c^2 = a^2 + h$. Then

$$c^{2p} - a^{2p} = (a^2 + h)^p - a^{2p} \equiv pa^{2(p-1)}h \pmod{h^2}.$$

Hence

$$k \equiv pa^{2(p-1)} \pmod{h}, \quad h = c^2 - a^2. \tag{3}$$

Now p does not divide h, since it does not divide b. Also a is relatively prime to h, since a and c have no common factor. Hence $pa^{2(p-1)}$ is relatively prime to h, so that k is relatively prime to h. Thus, in the factorization on the right of (2), the factors are relatively prime, and consequently each of them is a $2p$th power, and in particular is a square.

Each of a^{2p}, b^{2p}, c^{2p}, being a square, is congruent to 0 or 1 mod 3. Hence we can suppose that b is divisible by 3 and a, c are not. Hence $h = c^2 - a^2$ is divisible by 3, and so k is not.

The congruence (3) implies that

$$k \equiv pa^{2(p-1)} \pmod{3},$$

and since k is a square not divisible by 3, and $a^{2(p-1)}$ is a square not divisible by 3, it proves that $p \equiv 1 \pmod 3$.

A similar argument shows that $p \equiv 1 \pmod 8$. For we can suppose that b is even and a, c are odd, and since any odd square is $\equiv 1 \pmod 8$, it follows that

$$c^2 - a^2 = h \equiv 0 \pmod 8.$$

We conclude as before that $p \equiv 1 \pmod 8$.

Another similar argument shows that

$$p \equiv \pm 1 \pmod 5,$$

but here there are two possibilities. We can suppose that (1) b is divisible by 5 and a, c are not, or (2) c is divisible by 5 and a, b are not. In case 1 we have

$$c^2 \equiv \pm 1 \pmod 5 \text{ and } a^2 \equiv \pm 1 \pmod 5,$$

and since

$$c^{2p} - a^{2p} \equiv 0 \pmod 5,$$

we must have the same sign in both places. Hence

$$c^2 - a^2 = h \equiv 0 \pmod 5,$$

and it follows as before that $\pm 1 \equiv \pm p \pmod 5$, hence $p \equiv \pm 1 \pmod 5$.

In case 2, where c is divisible by 5 and a, b are not, we have to proceed a little differently. We have

$$c^{2p} = a^{2p} + b^{2p} = (a^2 + b^2)k',$$

where k' is a positive integer. In place of (3), we get

$$k' \equiv pa^{2(p-1)} \pmod{h'}, \qquad h' = a^2 + b^2.$$

As before, h' and k' are relatively prime, and each of them is a $2p$th power and so is a square. Since $h' = a^2 + b^2$ is divisible by 5, we have

$$k' \equiv pa^{2(p-1)} \pmod 5,$$

and since k' is a square not divisible by 5, it follows that $p \equiv \pm 1 \pmod 5$.

The three congruences with moduli 3, 8, 5 imply that $p \equiv 1$ or $49 \pmod{120}$.

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THE SEQUENCE OF TOUCHING CIRCLES

JOHN SATTERLY

This problem has been treated in many Geometries (Casey, Forder, Steiner) and in the Mathematical Gazette,[1] and by Frederick Soddy[2].

Perhaps I can add a little of interest.

Fig. 1 shows an outer semicircle of radius R and two inner semicircles of radii a and b such that $R = a + b$. Denote these semicircles by R, A, B, their centres by S, A, B respectively and the right hand end of the common diametrical line by O.

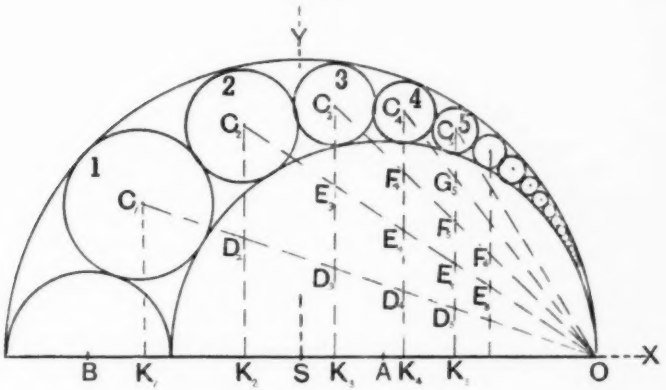


FIG. 1

A sequence of touching circles is inscribed in the region between the large semicircle and the two smaller semicircles. Number the circles in this sequence 1, 2, 3 - - - n - - - and denote their centres by C_1, C_2 - C_n -, their radii by c_1, c_2 - - c_n - - and the coordinates of these centres by $X_1 Y_1, X_2 Y_2$, - - - - - $X_n Y_n$ - - - these last being measured from axes SX, SY.

C_1. To get c_1 we see that it is the common intersection of three circles, viz a circle of centre S and radius $R - c_1$., a circle of centre A and radius $a + c_1$, and a circle of centre B and radius $b + c_1$.

[1] Mathematical Gazette A. P. Rollett Vol. 21, 1937, 412; Vol 22, 1938, 257; E. H. Neville Vol. 22, 1938, 288; A. N. Krishnaswami Ayyangar Vol. 22, 1938, 288; J. M. Child Vol. 32, 1948, 52; Richard Beetham, Vol. 37, 1953, 53.

[2] Frederick Soddy, "The Kiss Precise," Nature, Val. 137, 1936, 1021 and Vol. 139, 1937, 62.

The equations of these circles are

$$x^2 + y^2 = (R - c_1)^2$$
$$(x - R - a)^2 + y^2 = (a + c_1)^2$$
$$(x + a)^2 + y^2 = (b + c_1)^2$$

The solution of these equations gives

$$c_1 = Ra(R - a)/(R^2 - Ra + a^2) = Rab/(R^2 - Ra + a^2) \quad (1)$$
$$X_1 = R^2(R - 2a)/(R^2 - Ra + a^2) \quad (2)$$
$$Y_1 = 2Ra(R - a)/R^2 - Ra + a^2) = 2c_1 \quad (3)$$

The common denominator $R^2 - Ra + a^2$ may also be written $R^2 - Rb + b^2$, $Ra + b^2$, $Rb + a^2$, $a^2 + ab + b^2$, symmetrical in a and b. Having now found C_1, X_1, Y_1, intersecting circles may be used to find C_2, X_2, Y_2. The algebra however becomes complicated and it is easier to use Soddy's Theorem on Touching circles and its useful corollary which states that if a circle be described to touch three touching circles of radii a, b, c, its radius r is given by

$$r = abc/[(bc + ca + ab) \pm 2\{(abc)(a + b + c)\}^{\frac{1}{2}}] \quad (4)$$

If the positive sign of the square root is taken r is positive and it is the radius of the circle which lies within the common enclosure, and all contacts are interior contacts. If the negative sign of the square root is taken r is negative and it is the radius of the circle which encloses the three circles, i.e., the radius of the circle which is touched internally by the three circles: and must be taken as negative in calculations where if three of r, a, b, c are given the fourth is to be found.[3]

$\mathbf{C}_n$. By use of Eq. (4) with circles R, A, B we find

$$c_1 = Rab/(Ra + b^2) \text{ as obtained above.}$$

and by use of circles R, A, C_1, we find $c_2 = Rab/(Ra + 2^2b^2)$ and by use of circles R, A, C_2 we find $c_3 = Rab/(Ra + 3^2b^2)$ and so on. Therefore in general

$$c_n = Rab/(Ra + n^2b^2) \quad (5)$$

COROLLARY: If $a = b$ as in the case cited by Rollett $c_n = \dfrac{R}{n^2 + 2}$

$\mathbf{X}_n$. The easiest way to get X_n is to use two of the intersecting circles which locate c_n. Take those whose centres are S and A, their equations are

$$x^2 + y^2 = (R - c_n)^2$$
$$(x - b)^2 + y^2 = (a + c_n)^2$$

[3] Two integral solutions of Eq. (4) are (1) a, b, c equal to 23, 46, 69 respectively when the r's are 6 and -38, (2) a,b,c. equal to 15, 15, 10 when the r's are 2 and -30. A very near integral solution occurs when a, b, c are 8, 7 and 5 when the r's are 0.9976 .. and -15.0003 ..

The solution of these equations gives

$$X_n = R - \{(R + a)/b\}c_n = R(n^2b^2 - a^2)/(Ra + n^2b^2) \quad (6)$$

whence

$$R - X_n = \{(R + a)/b\}c_n = Ra(R + a)/(Ra + n^2b^2) \quad (7)$$

$\mathbf{Y}_n$. From $X_n^2 + Y_n^2 = (R - c_n)^2$ we find

$$Y_n = 2n\ Rab/(Ra + n^2b^2) = 2nc_n \quad (8)$$

The angle C_nOS

$$(R - X_n)/Y_n = \frac{1}{n}(R + a)/(2b) \quad (9)$$

or

$$\cot \angle C_nOS = \frac{1}{n}(R + a)/(2b) \quad (10)$$

Corollaries

(1) If $a = b = \frac{1}{2}R$ $\qquad Y_1 = Y_2 = \frac{2}{3}R$

If $a = 2b = \frac{2}{3}R$ $\qquad Y_2 = Y_3 = \frac{4}{5}R$

If $a = 3b = \frac{3}{4}R$ $\qquad Y_3 = Y_4 = \frac{5}{7}R$

or the "Curious Rectangle" of Rollett exists whenever a/b is integral. The left hand vertical side is always above S, the right hand vertical side is always above A. (Fig. 2).

(2) If $a = \frac{3}{2}b$ Eq. (10) gives the $\angle C_2OS = 45°$, if $a = \frac{5}{2}b$ the $\angle C_3OS = 45°$ (as in Fig. 1), if $a = \frac{7}{2}b$ the $\angle C_4OS = 45°$.

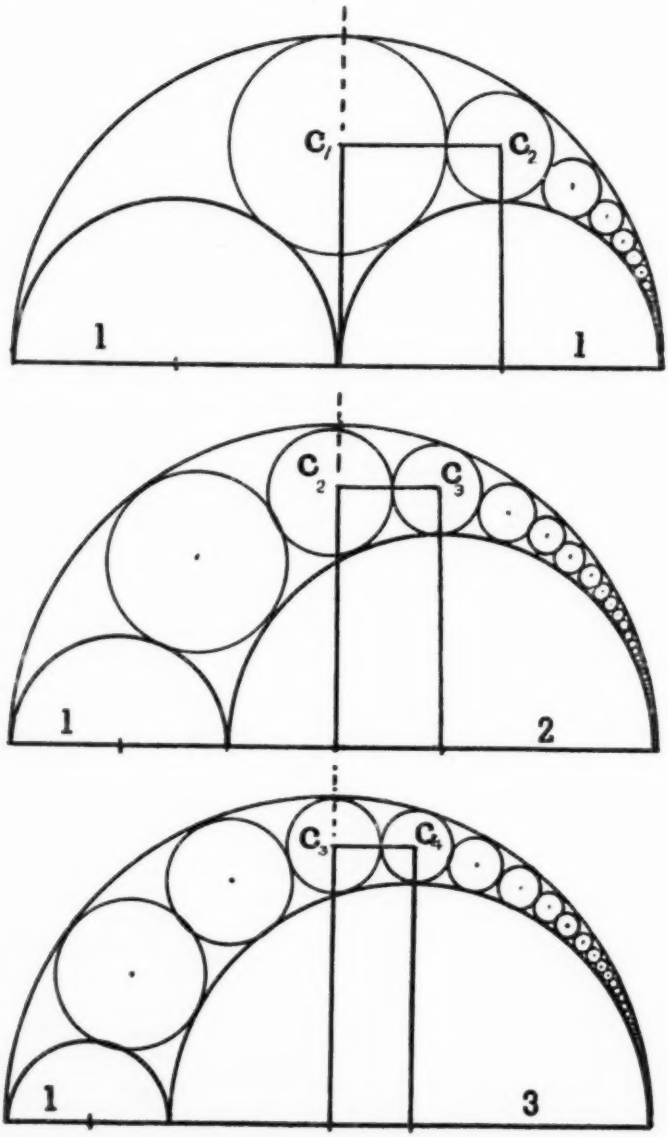
(3) If in Eqs. (5) to (10) we replace $Ra + n^2b^2$ by $Rb + n^2a^2$ we get the data for the sequence of circles which touch one another and the circles R and B.

One Numerical Example may be cited.

Radii of Circles: $R = 6, a = 4, b = 2$

Number of Circle	Name of Circle	Radius of Circle	Height of Centre, Y	Abscissa of Centre, X	$R - X$
0	B	12/6	0	24/6	60/6
1	C_1	12/7	24/7	18/7	60/7
2	C_2	12/10	48/10	0	60/10
3	C_3	12/15	72/15	30/15	60/15
4	C_4	12/22	96/22	72/22	60/22
5	C_5	12/31	120/31	126/31	60/31
6	C_6	12/42	144/42	192/42	60/42

Note the numerical sequences.



FIGS. 2. 3. AND 4

Intercept relations

Mere love of drawing led me to join C_1 to O, the line C_1, O cutting the ordinates of C_2, C_3, C_4 - - - in D_2, D_3, D_4 - - (Fig. 1). To my joy found $D_2K_2 = \frac{1}{2}C_2K_2$, $D_3K_3 = \frac{1}{3}C_3K_3$ - - - - i.e., the ordinates of the D's $= \dfrac{1}{n}$ (the ordinates of the C's). Similarly the join C_2O cuts the ordinates of C_3, C_4, - - - - - - in E_3, E_4 - - - - and the ordinates of the E's are $\dfrac{2}{n}$ (the ordinates of the C's) and the join C_3O cuts the ordinates of C_4, C_5 - - - - - in F_4, F_5 - - - - - such that the ordinates of the F's $= \dfrac{3}{n}$ (ordinates of the C's) and so on.

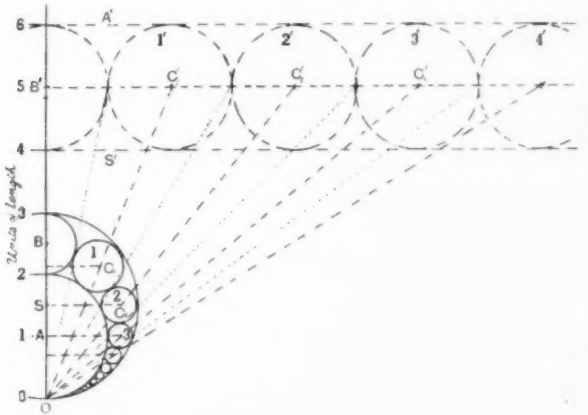


FIG. 5

A reference to *inversion methods* led me to draw Fig. 5 where the lower part of the figure shows the sequence of touching circles for the case $a = 2b = \frac{2}{3}R$. The centre of inversion is 0, the radius of inversion is $\sqrt{12}$ (see length scale at side), the dotted parallel lines A' and S' are the inverts of circles A and S and the equal dotted circles B', $1'$, $2'$, $3'$ are the inverts of circles B, C_1, C_2, C_3 - - - - - The centres B', C_1', C_2' C_3' are uniformly spaced and from this diagram the intercept relations given above are obvious.

A bit of history. When the above work was done I remembered that in December 1942 I received a letter from the late Professor Sir Charles V. Boys (of quartz fibre and soap bubbles fame and author of "The Natural Logarithm") in which he had referred to the problem of touching circles. In it he mentioned Pappus (*c* 500 B.C.) Ozanam of Paris (*c* 1700 A.D.) and Saggara and remarked

that he, Boys, had by Euclidean methods alone found the value of the radius of the first touching circle (my circle 1) and had confirmed the Pappus sequence relation ($Y_n = 2nc_n$). Also he remarked that the use of inversion methods gave simple solutions of much of the above problem.

J. S.

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GLEANINGS FAR AND NEAR

1948. In Vino Veritas:—"Si adunque moltiplicaremo la terza parte di *IM* lunghezza della Botte *BDFH* in due cerchi maggiore *CG* e uno de minore *BH*, *DF* come in *BH*, ci verrà la capacità di detta Botte." F. B. Cavalieri, *Una centuria di varii Problemi nella Prattica Astrologica*, (Bologna, 1639), Problema 80. [Per Dr. G. N. Watson. It was probably this formula for the capacity of a cask of wine which suggested to James Gregory (who spent the years 1665–8 in Padua) the more general formula for approximate integration, often written in the form

$$\int_0^{2h} y\,dx = h(y_0 + 4y_1 + y_2)/3,$$

which, in its turn, led to the outstanding generalisations due to Newton and Cotes, and subsequently to the minor generalisation known as 'Simpson's rule'; about this rule (published in 1743) there is not much to be said except that it is more accurate than the 'trapezoidal rule'.] [Per Dr. G. N. Watson]

1949. 1 in 5 of CSC Freshmen rank in top 10 PCT. of class.—[Headline in *Denver Post* quoted by New Yorker 9 Jan. 1960.] [Per Mr. C. A. B. Smith.]

1950. 'You can see from this greatly enlarged diagram [of a piece of mechanism] how small the original parts must be.'—Speaker in ITV school science programme. 24 June 1960. [Per Mr. R. F. Wheeler.]

1951. Without it [Thayer's *The Life of Beethoven*], we know nothing of Beethoven the man. The interpreters may be disregarded. We have Thayer, and we have the music. They are the two lines of a simultaneous equation, waiting for someone to reconcile and relate them. On this point I must disagree with Mr. Pryce-Jones: the two are not irreconcilable. It is merely that the problem is beyond our elementary algebra.—Colin Wilson in *The Sunday Times*, June 19, 1960. [Per Mr. F. Gerrish.]

2925. A property of 37

1. We consider numbers <1000 whose digits are x, y, z. If N_1 and N_2 are two such numbers which differ by a single transposition of two digits, the congruence $N_1 \equiv N_2 \text{ Mod } p$ gives

$$\text{For the transposition } (x, y) \ldots 90(x - y) \equiv 0 \text{ Mod } p$$
$$\text{For the transposition } (x, z) \ldots 99(x - z) \equiv 0 \text{ Mod } p$$
$$\text{For the transposition } (y, z) \ldots \ 9(y - z) \equiv 0 \text{ Mod } p$$

These congruences have no solutions if $p > 11$.

2. The 6 numbers formed by permutations of x, y, z can be arranged in two sets of 3 numbers such that no two numbers in a set are related by a single transposition. The two sets are:—

I $x, y, z;\quad y, z, x;\quad z, x, y.$ II $x, z, y;\quad y, x, z;\quad z, y, x$

Taking set I, we write the congruences:

$$100x + 10y + z \equiv 0 \text{ Mod } p \tag{1}$$
$$x + 100y + 10z \equiv 0 \text{ Mod } p \tag{2}$$
$$10x + y + 100z \equiv 0 \text{ Mod } p \tag{3}$$

The determinant of the coefficients is

$$D = \begin{vmatrix} 100 & 10 & 1 \\ 1 & 100 & 10 \\ 10 & 1 & 100 \end{vmatrix} = 998{,}001 = 3^6 \times 37^2$$

Taking set II instead of set I gives the same determinant. If we now put $p = 37$ in (1), (2), and (3) we have $D \equiv 0 \text{ Mod } 37$. Furthermore, the 2×2 minors of D are also all divisible by 37, so the Matrix of the coefficients is of rank 1, Modulo 37.

3. It follows that any set of values of x, y, z which satisfies one of the congruences (1), (2), (3) will also satisfy the other two. Such a set of values is given by the digits of any 3-figure multiple of 37. Thus any 3-figure multiple of 37 has two permutations of its digits which are also multiples of 37.

4. We can arrange the 3-figure multiples of 37 in sets of 3, as follows:

037,	370,	703	These sets are each
074,	407,	740	arithmetical progressions
148,	481,	814	with difference 333.
185,	518,	851	
259,	592,	925	
296,	629,	962	

The only other 3-figure multiples of 37 are the 9 numbers 111, 222, 333, ... 999.

5. It also follows that 37 is the only prime number >11 which has this property.

Rougham Hall, King's Lynn, Norfolk. ROGER NORTH

2926. Constructions possible by ruler and compasses

Using only ruler and compasses we are, as is well-known, unable to make the necessary constructions to solve the famous Greek problems of antiquity, namely, the duplication of a cube, the squaring of a circle, and the trisection of an angle. An immediate consequence of this failure is the question: Just what constructions are possible using ruler and compasses, either separately or together? It is the purpose of this article to answer this query by outlining briefly the history of the constructions. The answer may be organized under four headings:

1. Constructions using ruler and compasses;
2. Constructions using compasses only;
3. Constructions using a ruler only;
4. Constructions using other related instruments.

When ruler and compasses are restricted to the use intended of them by the Greeks, we shall call them *Euclidean Instruments*, and methods of construction thus permissible will be called *Euclidean methods* (whether or not Euclid employed them). Such usage is limited by the Postulates laid down in the *Elements* (Heath [1]):

"Let the following be postulated:

(i) To draw a straight line from any point to any point;

(ii) To produce a finite straight line continuously in a straight line;

(iii) To describe a circle with any centre and distance" (i.e., radius).

Observe that nothing is said about the size of the circle: it may be infinitely small or infinitely large.

From the first two Postulates, we see that the function of the ruler is merely to draw and produce straight lines, i.e., it is ungraduated and must not be used as an instrument for measuring lengths—in this respect, it may more appropriately be called a *straight edge*. Postulate (iii) limits the power of compasses: they do not carry distance except in so far as all radii of the same circle are of equal length.

Here we must distinguish clearly between the Euclidean compasses, an instrument capable only of describing a circle, and modern compasses which not only possess this power but also possess the ability to transfer measurements, i.e., to behave like a pair of dividers, which is quite a different instrument.

Another clear distinction must be made between Physical Geometry whose raw materials are physical objects such as points, lines, and circles, and Abstract (or Mathematical) Geometry where the objects bearing these names are idealizations of their physical counterparts. Euclid's Postulates deal with abstract, conceptual objects, but they imply the existence of physical instruments capable of producing practical representations of the ideal material of the *Elements*. Of course, the Greeks were very much aware of this twofold aspect of Geometry. Aristotle, for instance, remarks that the geometer's proof concerns ideal elements of which the physical representations are imperfections. With reference to the first Postulate Simplicius observes that it allows a mathematical straight line to be imagined rather than to be physically drawn. For, says Simplicius: "he would be a rash person who, taking things as they actually are, should postulate the drawing of a straight line from Aries to Libra."

Before proceeding, it is interesting to speculate on two questions: What compasses were available to the Greeks, i.e., what mechanical device had Euclid in mind for constructing the circle (possibly drawn on the sand)? And: Why did the Greeks, as in Euclid's Postulates, limit their geometrical apparatus to ruler and compasses?

Aristophanes in *The Clouds*, which was performed at the Great Dionysia in 423 B.C., makes the Disciple of Socrates tell Strepsiades that Socrates, who is being lampooned, "took up a pair of compasses ..." (*diabetes*), so that the Greeks were familiar with this instrument by the fifth century before Christ. Euclid's own definition (Def. 17) of a circle as a "plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another" says nothing about the actual construction, which is postulated as Postulate (iii). On the other hand, Simplicius argues that Definition 17 implies that Euclid intended the circle to be constructed by the revolution of a line about one end, the centre. Commenting on the limitations placed on Euclid by his third Postulate, Hudson [2] observes: "... his instrument, whatever it was, must have collapsed in some way as soon as the centre was shifted, or either point left the plane."

Etruscan compasses of a primitive kind, somewhat like our dividers in appearance, have been unearthed by archaeologists, and proportional Roman compasses, divided in the ratio 1:3, have come to light at Pompeii. Perhaps the compasses Euclid was familiar with resembled either of these.

Part of our answer to the second query may be found in Plato's strong aversion to *mechanical* solutions (which involved conics and higher plane curves*) as opposed to *geometrical* solutions involving

* Constructible by various mechanical devices.

only ruler and compasses. According to one tradition it was Plato's influence which confined constructions to these two instruments, though Heath hazards the guess that this may have been due to the pre-Platonic geometer and astronomer, Oenopides of Chios. Says Plato: "... the good of geometry is set aside and destroyed, for we again reduce it to the world of sense instead of elevating and imbuing it with the eternal and incorporeal images of thought ...."

Isaac Newton gives the following answer: "It is not the simplicity of the equation, but the easiness of the description, which is to determine the choice of our lines for the constructions of problems. For the equation that expresses a parabola is more simple than that that expresses a circle, and yet the circle, by its more simple construction, is admitted before it." Nor must it be overlooked that the circle, because of its "perfection," had in earlier times exercised a mystical power over human thinking.

1. *Constructions Using a Ruler and Compasses*

Basically, problems concerning constructions can be reduced to the determination of points satisfying given conditions. Euclidean modes of determination allow us to construct a point in any of the three following ways: (a) as the intersection of a pair of straight lines; (b) as the intersection of a circle and a straight line; (c) as the intersection of two circles. In (c), one of the circles may be replaced by the common chord of the two circles and, consequently, (c) is equivalent to (b).

Modern methods in algebra and analysis have been employed to elicit the types of construction possible using only ruler and compasses together, or separately. It has been found that only when we know the analytical processes which are equivalent to each geometrical step are we able to say what constructions may be made using the given instruments in a given way. What then are the possible constructions with ruler and compasses? The answer is (Hudson [2]): Only those problems can be solved by ruler and compasses which depend on an algebraic equation whose degree is a power of 2 and whose roots are found by performing (a finite number of times) rational operations together with the extraction of square roots.

Not only did this important result lay the ghost which had haunted the ancient Greeks, but it also answered the question: Which regular polygons can be constructed by ruler and compasses? Gauss, with the help of number theory, showed in 1796 (as an entry in his diary reveals) that polygons can be constructed if the number of their sides is $n = 2^{2^p} + 1$ (i.e., $n - 1$ is a power of 2), provided that n is a prime number. Otherwise, if n is not a prime number, the construction is impossible. Thus, for $p = 2$, Gauss demonstrated

that a regular polygon of 17 sides (a heptadecagon) could be constructed by Euclidean methods. (Previously, only regular polygons with sides numbering 2^k, 3, 5 and products of these, were determinable by Euclidean techniques.) The first purely geometrical construction of the regular heptadecagon was given in 1825 by Erchinger, and was known to Gauss. Alternatively, the various steps of the construction may be made to parallel the algebraic steps, as in Klein [3]. For a relatively short construction using purely geometrical considerations, one may refer to that of Richmond, quoted in Hudson [2]. As a student at Göttingen University, Gauss was undecided, whether to specialize in mathematics or ancient languages, but his discovery finally made up his mind for him, and what a tremendous gain for the world of mathematics this turned out to be. Immensely proud of his discovery, Gauss requested that the heptadecagon be engraved on his tombstone. Although this was not done, the heptadecagon was engraved on a monument which was erected to honour Gauss in the town of his birth, Braunschweig.

Next in the hierarchy of Gaussian regular polygons constructible by Euclidean methods are those whose sides number 257 ($p = 3$), and 65,537 ($p = 4$) on which Professor Hermes of Lingen is reputed to have laboured for 10 years, "examining with care all the roots furnished by Gauss' method." (Klein [3]). Beyond $p = 4$, the values of n are not known to be prime.

Attempts to perform the impossible Greek constructions by Euclidean methods have often been ingenious and amazingly accurate. Ramanujan, for instance, was able to give in 1913 a ruler and compasses construction for an approximation to π which was so near the mark that if the area of the circle had been 140,000 square miles his length would have been incorrect by about 1 inch.

2. *Constructions Using Compasses Alone*

By means of inversion, it may be shown (e.g., Hobson [4]) that (a) and (b) involve (c), i.e., that any point occurring in a Euclidean problem can be constructed as an intersection of two circles. This implies that, in Physical Geometry, the compasses alone are sufficient for all constructions with ruler and compasses. That every problem which can be solved with ruler and compasses can also be solved by compasses alone was first established in 1672 by the Danish mathematician, Georg Mohr. Usually, however, Lorenzo Mascheroni, who published his *Geometria del Compasso* in 1797, is credited with this discovery.

Because a ruler may be warped or in some other respect inadequate as a straight edge, Mascheroni's constructions with compasses were claimed to yield very accurate determinations. On this account they were beneficial to instrument makers and, in particular, to

makers of telescopes where great precision is required in graduating reading circles.

Mascheroni's original treatment of constructions with compasses was considerably improved by the use of inversion in 1890 by Adler. (See Hobson [4].) Moreover, Adler's use of compasses was much purer. Whereas Mascheroni sometimes employed his compasses (which were not, of course, restricted to a fixed radius) as dividers, i.e., to carry distance, Adler used only Euclidean compasses. Mascheroni's researches attracted the attention of Napoleon who posed the following problem to French mathematicians: Divide the circumference of a circle into four equal parts using compasses only.

Compasses can never completely perform the function required of Euclid's first two postulates, i.e., no straight line can be entirely obtained by using only compasses. When two points have been constructed, we have to regard the line determined by these two points as known or imagined. Otherwise, we could use the ruler merely for joining pairs of points, i.e., for completing (e.g. dotting in) lines of the figure, but this does not, as Cayley [5] remarks, "in any wise interfere with the fundamental Postulate that the constructions are to be performed with the compass (sic) only."

3. *Ruler Constructions*

Perhaps the most interesting, because of its simplicity, of the instruments of Euclidean constructions is the ruler. (Because it is a non-metrical agent, the Euclidean ruler is eminently suited to projective constructions.) Firstly, we consider the use of the ruler on its own. It may be shown (Hudson [2]) that the only problems which can be solved entirely by ruler constructions are those expressible as a linear equation, the roots of which can be evaluated using purely rational operations.

Mathematicians who effected constructions with a ruler alone were, inter alia, Lambert (1774), Servois (1805), and Brianchon, who, by 1818, was quite successful in achieving his aim. Like Mascheroni, Brianchon insisted on the practical significance of his geometrical constructions: in his case, they were claimed to have value in surveying.

Use of the ruler was cleverly exploited and extended by Poncelet who, in 1882, was the first to realize that, with a circle and its centre given, every solution of a problem by means of ruler and compasses can be achieved with a ruler alone. This is sometimes stated as follows: every construction with ruler and compasses can be effected with a ruler, and compasses with a fixed opening. Constructions of this sort are traceable to the Persian geometer Abû'l Wefâ (940–998), famous also as an astronomer (in particular, for his discovery of the variation of the Moon). Although Pappus

once restricted himself in this way, it was Abû'l Wefâ who repeatedly performed constructions with compasses of a fixed opening. Occasionally both Leonardo da Vinci and Albrecht Dürer used this type of construction, and by the sixteenth century Tartaglia and Benedetti, to mention but a few Italian geometers, were using it freely. (It may be noted that Poncelet's geometrical work was carried out while he was a prisoner of the Russians at Saratov during Napoleon's disastrous march to Moscow.)

Eleven years later, in 1833, Steiner [6], who was fanatically keen on pure (synthetic) geometry, developed Poncelet's ideas in a remarkable book whose aim was "to solve all geometrical problems by means of the ruler alone, if any fixed circle [and its centre] (Archibald's correction) is given in the plane." Elsewhere in the text, he says that "it turns out that this objective may be more easily attained than I at first thought possible and than appeared to be possible, considering the nature of the subject." Steiner claimed that his constructions would be valuable to engineers and surveyors. Klein [3], following von Staudt (1842) and Schröter (1872), used the spirit of the Poncelet-Steiner ideas in his constructions of the regular heptadecagon.

Important developments affecting ruler constructions were made in 1890 by Adler who replaced the Poncelet-Steiner circle by a given central conic and one of its foci. Adler argued that constructions with ruler and compasses could then be performed by ruler alone, and that every problem involving equations of the third and fourth degree could be solved by ruler and compasses. In 1910, Morduchaï-Boltovskoï showed that any point constructible with ruler and compasses could be determined with ruler alone provided we are given an arc (however small) of a circle, and its centre.

Much of the work of Adler and Morduchaï-Boltovskoï was anticipated by H. J. S. Smith in 1869. Among other things he had generalized Morduchaï-Boltovskoï's arc of a circle (however small) to arc of a conic. Another result of Smith's was to substitute for the Poncelet-Steiner circle a Cissoid of Diocles, by means of which he showed that problems depending on equations of the third and fourth degree could be solved by a ruler.

Other pioneer researchers in the field of ruler constructions were Kortum, Ferrari, Cauer, and Grossmann, to name only a few.

4. *Constructions Using Other Related Instruments*

Elementary instruments such as dividers and two-edged rulers have also been used to carry out Euclidean operations, sometimes more quickly than can be achieved with ruler and compasses.

Using ruler and dividers, which can transfer distance, it can be shown that all constructions possible by these means can be achieved by ruler and compasses, though perhaps not so neatly. However,

the converse is not valid. Ruler and dividers, though obviously more efficient than just a ruler, are found to be more limited in their scope than ruler and (Euclidean) compasses (Hudson [2]).

If we have two rulers rigidly connected, a new instrument called a *two-edged ruler* is obtained. When the angle between the two edges is a right-angle, the instrument so formed is a T-square, or, as a variation of this, the carpenter's square. If the two edges meet at an angle which is not a right-angle we obtain a set-square. Finally, if the two edges are parallel, i.e., do not meet, so that the angle between them is zero, we have a parallel ruler.

Poncelet was the first to demonstrate that every point which can be constructed with a ruler and compasses can be constructed with a two-edged ruler alone, whether the edges are parallel or meet. Two carpenter's squares are known to produce the solutions of cubic equations and, in particular, of those equations connected with the Greek problems of duplicating a cube and trisecting an angle.

Paper folding has also been a technique by which the construction of points has been used to advantage. Both Sundara Row in India and Hermann Wiener in Germany experimented independently in this way in 1893. Sundara Row, for instance, used folding paper to construct any points on such diverse curves as ellipses and Cissoids.

Authoritative guides to the rather extensive literature on the above subjects may be found in Archibald's historical comments in Klein [3] and Steiner [6], and elsewhere in Klein.

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2927. A method of calculating some natural logarithms

Let

$$a = -\log\left(1 - \frac{1}{10}\right), b = -\log\left(1 - \frac{4}{100}\right), c = \log\left(1 + \frac{1}{80}\right),$$

$$d = \log\left(1 + \frac{8}{1000}\right), e = -\log\left(1 - \frac{1}{100}\right), f = \log\left(1 + \frac{1}{90}\right).$$

Then the results

$$\log 2 = 7a - 2b + 3c$$
$$\log 3 = 11a - 3b + 5c$$
$$\log 5 = 16a - 4b + 7c$$
$$\log 7 = 19a - 4b + 8c + d$$

are due to J. C. Adams. They may all be proved by the same simple method which we use to obtain a similar result for log 11.

Let

$$11 = \left(\frac{10}{9}\right)^k \cdot \left(\frac{25}{24}\right)^l \cdot \left(\frac{81}{80}\right)^m \cdot \left(\frac{126}{125}\right)^n \cdot \left(\frac{100}{99}\right)^p$$

$$= 2^r \cdot 3^s \cdot 5^t \cdot 7^u \cdot 11^{-p}$$

where
$$r = k - 3l - 4m + n + 2p$$
$$s = -2k - l + 4m + 2n - 2p$$
$$t = k + 2l - m - 3n + 2p$$
$\therefore$ $p = -1, \quad r = s = t = u = 0$
$\therefore$ $k = 24, \quad l = -6, \quad m = 10$
$\therefore$ $\log 11 = 24a - 6b + 10c - e.$

Similarly

$$\log 13 = 26a - 8b + 12c - d + f.$$

The values of a, b, c, d, e, f may be quickly calculated to several decimal places by using the expansion in powers of x of $\log(1 + x)$, $|x| < 1$.

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2928. The calculation of common logarithms

1. It is generally said that tables of logarithms are constructed using the logarithmic series to find natural logarithms and then changing the base. Of course this is not the only way; and in fact much might be done by interested students long before they know anything about natural logarithms. The following material could be used as a project for better students when they are learning the index laws and the definition of logarithm; or it might replace some

of the customary exercises on indices. Such exercises often look artificial, even to the teacher; the alternative offered below might be more effective since it has a clear motive.

For a start, rough approximations to the logarithms of some simple numbers can be found very easily:

$$2^{10} = 1024 \simeq 1000 = 10^3, \qquad 2 \simeq 10^{3/10}, \qquad \log 2 \simeq 0{\cdot}30.$$

$$3^4 = 81 \simeq 80 = 2^3 \times 10 \simeq 10^{3\times 0{\cdot}3+1} = 10^{1{\cdot}9},$$

$$3 \simeq 10^{1{\cdot}9/4}, \qquad \log 3 \simeq 0{\cdot}48.$$

$$5 = 10 \div 2 \simeq 10^{1-0{\cdot}3} = 10^{0{\cdot}7}, \qquad \log 5 \simeq 0{\cdot}70.$$

$$7^2 = 49 \simeq 50 = 5 \times 10 \simeq 10^{0{\cdot}7+1} = 10^{1{\cdot}7},$$

$$7 \simeq 10^{1{\cdot}7/2}, \qquad \log 7 \simeq 0{\cdot}85.$$

From these, rough logarithms of all composite numbers having no prime factor greater than 7 are immediately obtainable. Many of the entries in the first column of the logarithm table can be simulated in this way; for instance,

$$4 = 2^2 \simeq (10^{0{\cdot}3})^2 = 10^{0{\cdot}6}, \qquad \log 4 \simeq 0{\cdot}60.$$

$$5{\cdot}6 = \frac{2^3 \times 7}{10} \simeq 10^{0{\cdot}9+0{\cdot}85-1} = 10^{0{\cdot}75}, \quad \log 5{\cdot}6 \simeq 0{\cdot}75.$$

The role of the prime numbers in this will soon be perceived. Opportunities for the use of imagination arise in seeking factorizations which will permit successive prime numbers to be expressed as powers of 10. In so doing, familiarity with numbers will be developing. Here are a few instances:

$$11^2 = 121 \simeq 2^2 \times 3 \times 10, \qquad 11^3 = 1331 \simeq (4/3) \times 10^3,$$

$$13^2 = 169 \simeq 2^3 \times 3 \times 7, \qquad 13^3 = 2197 \simeq 2 \times 11 \times 10^2.$$

2. A year or two later, similar work to a higher degree of accuracy might be encouraged. It could serve to remind students of the meaning of logarithm, and the dependence of logarithms on the index laws, at a time when continual use and familiarity have tended to obscure these important things. Knowledge of the approximation $(1+x)^n \simeq 1+nx$ when n is a positive integer and x is small might be an advantage now; but it would not be essential. Maturity would be needed, rather than extra knowledge.

The previous calculation of log 2 was based on $2^{10}/10^3$ being close to 1. It is actually 1·024, and we refine that result by expressing 1·024 as a power of 10. As preliminaries we verify that

$$1{\cdot}024^4 \simeq 1{\cdot}0995 \simeq 1{\cdot}100, \tag{1}$$

$$1{\cdot}024^{12} \simeq 1{\cdot}100^3 = 1{\cdot}331.$$

$$1{\cdot}024^{96} \simeq 1{\cdot}331^8 \simeq 9{\cdot}86. \simeq 10. \tag{2}$$

So
$$2^{10}/10^3 = 1{\cdot}024 \simeq 10^{1/96}, \tag{3}$$

and
$$2 \simeq 10^{289/960} \simeq 10^{0{\cdot}30104}. \tag{4}$$

The accuracy obtained here is less accidental than it looks. The approximations leading to (2) show that the error in replacing $1{\cdot}024^{96}$ by 10 is about 3%. Taking the 96th root, the error in (3) is about 0·03%; and that in (4) about 0·003%.

For log 3 we may use an approximation already implicit above:

$$10^{1/8} \simeq 1{\cdot}3335 \simeq \tfrac{4}{3}. \tag{5}$$

giving $$3 \simeq 2^2 \times 10^{-1/8} \simeq 10^{0{\cdot}6021-0{\cdot}1250} = 10^{0{\cdot}4771}.$$

Again $$7^4 = 2401 \simeq 2^3 \times 3 \times 10^2 \simeq 10^{3{\cdot}3802},$$

so that $$7 \simeq 10^{0{\cdot}8451}.$$

3. To continue, we search for approximate equalities between composite numbers, relating each prime number with those preceding it (and with 10) in a way involving a small enough fractional error. Instances are:

$$1001 \simeq 1000, \quad 1700 \simeq 1701, \quad 4199 \simeq 4200, \quad 1332 \simeq 1331.$$

For log 11 we may use

$$\left(\frac{11 \times 9}{10^2}\right)^4 = \left(1 - \frac{1}{100}\right)^4 \simeq 1 - \frac{4}{100} = \frac{2^5 \times 3}{10^2}, \tag{6}$$

giving $$11^4 \simeq 2^5 \times 10^6/3^7,$$

from which $$11 \simeq 10^{1{\cdot}0114}.$$

A similar procedure may be used for 13, 17, 19, ... Indeed, (6) can be generalized to a formula expressing $2n + 1$ as a product of powers of $2n - 1$, $n + 1$, n, and $n - 1$ with fractional error $O(1/n^4)$. But when I found this formula I lost interest.

Another method for log 13 is

$$\frac{13^3}{11^3} = \frac{2197}{1331} = \frac{2197(1 + \frac{3}{2196})}{1331(1 + \frac{2}{1464})} \simeq \frac{2200}{1333\frac{1}{3}} = \frac{2{\cdot}2}{4/3} = \frac{11 \times 3}{10 \times 2},$$

giving $$13^3 \simeq 3 \times 11^4/2 \times 10,$$

from which $$13 \simeq 10^{1{\cdot}1139}.$$

4. The whole scheme leans heavily on (5). Of course, that approximation is nothing but a fortunate accident; but it is not essential. We could start quite differently, writing

$$2 = 10^x, \quad 3 = 10^y, \quad 7 = 10^z$$

in the approximate equalities	and obtain simultaneous equations
$2^3 \times 3 \times 10^2 = 7^4$	$3x + y - 4z = -2$
$2^5 \times 3^7 = 7 \times 10^4$	$5x + 7y - z = 4$
$2^{18} \times 7^3 = 3^2 \times 10^7$.	$18x - 2y + 3z = 7$.

These give the correct four-figure values for y and z, and a value for x in which the error increases to 1 in the fourth place on rounding off.

5. The 3% error mentioned in (2) suggests a possibility of improvement, which can be achieved by less rounding off of $1{\cdot}024^4$ in (1):

$$1{\cdot}024^{12} \simeq (1{\cdot}1 - 0{\cdot}0005)^3 \simeq 1{\cdot}3292, \quad 1{\cdot}024^{96} \simeq 1{\cdot}3292^8 \simeq 9{\cdot}744,$$
$$1{\cdot}024^{97} \simeq 9{\cdot}978, \simeq 10.$$

So, in place of (3) and (4) we have

$$2^{10} \simeq 10^3 \times 10^{1/97}, \qquad 2 \simeq 10^{292/970} \simeq 10^{0{\cdot}30103}.$$

Still greater accuracy can be reached by extrapolating from the above 96th and 97th powers of 1·024; we obtain $1{\cdot}024^{97{\cdot}1} \simeq 10$, and this gives log 2 correct to six decimal places.

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2929. On the error in numerical integration

If $f(x)$ is defined in the interval $-1 \leq x \leq 1$, and has derivatives up to the $2n$th order, with its $2n$th derivative continuous, then by the generalized mean value theorem,

$$f(x) = \sum_{r=0}^{2n-1} \frac{f^{(n)}(0)}{r!}\, x^r + \int_0^x \frac{(x-t)^{2n-1}}{(2n-1)!}\, f^{(2n)}(t)\, dt \qquad (1)$$

Let
$$A = \sum_{s=1}^{p} \xi_s (f(\alpha_s) + f(-\alpha_s)) \qquad (2)$$

where
$$\sum_{s=1}^{p} \xi_s \alpha^{2r} = \frac{1}{2r+1}, \qquad r = 0, 1, 2, \ldots, 2n-1 \qquad (3)$$

then if $f(x)$ is a polynomial of degree not higher than $2n-1$, A gives the value of $\int_{-1}^{1} f(x)\, dx$. In general A is an approximation to the value of this integral.

In this note we find an expression for the error when A is taken as this integral and f is of the form (1). From this expression we derive the well known result for the maximum error when A is obtained by Simpson's rule. We also find the maximum errors when A is found by Tchebycheff's rule and a rule given by S. J. Tupper. These three results are all "best possible."

From (1) we find

$$\int_{-1}^{1} f(x)\, dx = 2\sum_{r=0}^{n-1} \frac{f^{2r}(0)}{(2r+1)!} + \int_{-1}^{1} dx \int_0^x \frac{(x-t)^{2n-1}}{(2n-1)!} f^{(2n)}(t)\, dt; \quad (4)$$

on changing the order of integration, the second term on the right becomes

$$\int_0^1 \frac{(1-t)^{2n}}{(2n)!} \{f^{(2n)}(t) + f^{(2n)}(-t)\}\, dt. \qquad (5)$$

again from (2) and (1),

$$A = 2\sum_{s=1}^{p}\sum_{r=1}^{n-1}\xi_s\alpha_s^{2r}\frac{f^{(2r)}(0)}{(2r)!} + \sum_{s=1}^{p}\int_0^{\alpha_s}\frac{\xi_s(\alpha_s-t)^{2n-1}}{(2n-1)!}\{f(t)+f(-t)\}\,dt \tag{6}$$

Using (3), the first term reduces to $2\sum_{r=0}^{n-1}\frac{f^{(2r)}(0)}{(2r+1)!}$.

If we now define the function $e_s(t)$ by $e_s(t) = 1$ for $t \leqq \alpha_s$ and $e_s(t) = 0$ when $\alpha_s < t$, and set

$$\phi(t) = \sum_{s=1}^{p}\frac{e_s\xi_s(\alpha_s-t)^{2n-1}}{(2n-1)!} - \frac{(1-t)^{2n}}{(2n)!} \tag{7}$$

then if E is the error in taking A as the integral, we may write $E = A - \int_{-1}^{1} f(x)\,dx$. From (4), (5), (6), (7)

$$E = \int_0^1 \phi(t)\{f^{(2n)}(t) + f^{(2n)}(-t)\}\,dt. \tag{8}$$

If $M = \max|f^{(2n)}(t)|$ for $-1 \leqq t \leqq 1$, then $|E| \leqq 2M\int_0^1|\phi(t)|\,dt$ (9)

If $\phi(t)$ is of constant sign we may replace (9) by

$$|E| \leqq 2M\left|\int_0^1\phi(t)\,dt\right| \tag{10}$$

This formula also applies in the case where $f^{(2n)}(t)$ is a constant, i.e. $f(t)$ is a polynomial of degree n, when (10) gives the exact error. Hence (10) is a "best possible" result when $\phi(t)$ is of constant sign. From (7), in this case,

$$|E| \leqq 2M\left|\sum_{s=1}^{p}\frac{\xi_s\alpha_s^{2n}}{(2n)!} - \frac{1}{(2n+1)!}\right| \tag{11}$$

Applying this result to Simpson's rule, we have

$$A = \tfrac{1}{3}(4f(0) + f(1) + f(-1)).$$

A is exact if $f(x)$ is a polynomial of degree three. We take $\alpha_1 = 0$, $\alpha_2 = 1$ and $n = 2$. Then

$$\phi(t) = \frac{1}{3}\frac{(1-t)^3}{3!} - \frac{(1-t)^3}{4!}$$

and is clearly of constant sign for $0 \leqq t \leqq 1$. Hence (11) applies and we get

$$|E| \leqq 2M\left(\frac{1}{3(4!)} - \frac{1}{5!}\right) = M/90.$$

This result is well known.

In Tchebycheff's rule

$$A = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right).$$

It is correct if $f(x)$ is of degree no higher than three. Again $n = 2$. We have

$$\phi(t) = \frac{\left(\frac{1}{\sqrt{3}} - t\right)^3}{3!} - \frac{(1-t)^4}{4!}$$

We will prove that $\phi(t)$ is of constant sign in $0 \leq t \leq 1$.
When $t = 0$,

$$\phi(t) = \frac{1}{18\sqrt{3}} - \frac{1}{24}$$

and is negative. Obviously

$$\phi(t) < 0 \quad \text{for} \quad \frac{1}{\sqrt{3}} \leq t < 1.$$

It will thus be sufficient to prove that

$$\phi(t) \neq 0 \quad \text{for} \quad 0 < t < \frac{1}{\sqrt{3}}.$$

If $\phi = 0$, then

$$\frac{\left(\frac{1}{\sqrt{3}} - t\right)^3}{3!} = \frac{(1-t)^4}{4!}$$

and so

$$\log \frac{\left(\frac{1}{\sqrt{3}} - t\right)^3}{3!} = \log \frac{(1-t)^4}{4!}.$$

Let

$$\psi = \log \left(\frac{1}{\sqrt{3}} - t\right)^3 \Big/ 3! - \log (1-t)^4/4!.$$

Then

$$\psi' = \frac{-3}{\frac{1}{\sqrt{3}} - t} + \frac{4}{1-t}.$$

Hence $\psi' < 0$ in $0 < t < \frac{1}{\sqrt{3}}$. Since $\psi < 0$ when $t = 0$, $\psi \neq 0$ in this range and hence $\phi \neq 0$. We may thus use (11) and get

$$|E| \leq 2M \left| \frac{1}{9 \cdot 4!} - \frac{1}{5!} \right| = \frac{M}{135}.$$

S. J. Tupper's formula is obtained by combining Simpson's rule and Tchebycheff's (see note 2531). It may be written as

$$A = \frac{1}{15}\left\{8f(0) + 9\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right] + 2[f(1) + f(-1)]\right\}$$

It is exact for a polynomial of degree not higher than 5. In this case

$$\phi(t) = \frac{3}{5} \cdot \frac{1}{5!}\left(\frac{1}{\sqrt{3}} - t\right)^5 + \frac{2}{15} \cdot \frac{1}{5!}(1-t)^5 - \frac{(1-t)^6}{6!}$$

$$\text{for } 0 \leqq t \leqq 1/\sqrt{3}$$

and

$$\phi(t) = \frac{2}{15} \cdot \frac{1}{5!}(1-t)^5 - \frac{(1-t)^6}{6!} \qquad \text{in } \frac{1}{\sqrt{3}} < t \leq 1.$$

The last expression can be put in the form

$$\phi(t) = -\frac{(1-t)^5(1-5t)}{5 \cdot 6!} \qquad \frac{1}{\sqrt{3}} < t \leq 1$$

which is positive when $1/5 < t$. Hence $0 \leq \phi(t)$ when $1/5 \leq t$.

Using the same technique we used in Tchebycheff's formula, we write

$$\psi(t) = \log \frac{3\left(\frac{1}{\sqrt{3}} - t\right)^5}{5 \cdot 5!} - \log \frac{(1-t)^5(1-5t)}{5 \cdot 6!}.$$

We get

$$\psi'(t) = -\frac{5}{\frac{1}{\sqrt{3}} - t} + \frac{5}{1-t} + \frac{5}{1-5t}$$

$$= \frac{5\left[t^2 + \left(4 - \frac{6}{\sqrt{3}}\right)t + \frac{2}{\sqrt{3}} - 1\right]}{\left(\frac{1}{\sqrt{3}} - t\right)(1-t)(1-5t)}$$

Hence $0 < \psi'$ in $0 < t < 1/5$, and since $0 < \psi(0)$, $0 < \psi(t)$ for $0 \leq t \leq 1/5$. Thus $0 \leq \psi$ for $0 \leq t \leq 1$. Thus formula (11) applies and we get

$$|E| \leq 2M\left(\frac{3}{5}\frac{1}{27 \cdot 6!} + \frac{2}{15} \cdot \frac{1}{6!} - \frac{1}{7!}\right) = M/28350,$$

where

$$M = \max |f^6(t)| \qquad \text{in } -1 \leq t \leq 1.$$

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2930. On note 2874

The proof of the well known theorem about even perfect numbers published in Note 2874 (*Math. Gaz.* XLIV No. 347, February 1960, p. 45) is also well known and can be found in many textbooks. I may quote only I. V. Arnold. *Theory of numbers.* Moscow, 1939, p. 73 or N. Obrechkoff. *Theory of numbers.* Sofia, 1954, p. 29.

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2931. On note 2853

May we draw the attention of your readers to an omission in a recent note by one of us (Note 2853, M.G. No. 345, pp. 188–9). It was there shown that if $f(x) = a_0x^m + \ldots + a_m$, $g(x) = b_0x^n + \ldots + b_n$, and $\mathbf{R}(y)$ is the $(m + n) \times (m + n)$ matrix

$$\begin{pmatrix} a_0 & a_1 \ldots a_m - y & & \\ & a_0 & a_1 \ldots a_m - y & \\ & & \ddots & \\ & & a_0 & a_1 \ldots a_m - y \\ b_0 & b_1 \ldots b_n & & \\ & b_0 & b_1 \ldots b_n & \\ & & \ddots & \\ & & b_0 & b_1 \ldots b_n \end{pmatrix}$$

(all elements not written being 0) then $|\mathbf{R}(y)| = 0$ for $y = y_r = f(x_r)$, where $x_1, \ldots, x_n$ are the zeros of $g(x)$. It was then stated that if $|\mathbf{R}(0)| = 0$, one of the y_r must be zero, thus proving that $|\mathbf{R}(0)| = 0$ is a sufficient condition for $f(x)$ and $g(x)$ to have a common zero. This argument, however, is valid as it stands only if the y_r are all different.

To complete the proof let $z_1, \ldots, z_m$ be the zeros of $f(x)$, so that $y_r = a_0(x_r - z_1) \ldots (x_r - z_m)$; and first suppose that the x's and z's are all unequal. Since $(-1)^{mn}b_0^{-m}|\mathbf{R}(0)|$ is the product of the roots of the equation $|\mathbf{R}(y)| = 0$, $|\mathbf{R}(0)|$ is divisible by $x_r - z_1$ and, by symmetry, by every difference $x_r - z_s$. Since $|\mathbf{R}(0)|$ is a polynomial in $a_0, b_0, x_1, \ldots, z_m$, consideration of degree gives

$$|\mathbf{R}(0)| = (-1)^{mn}a_0^{\,n}b_0^{\,m}(x_1 - z_1) \ldots (x_n - z_m);$$

and this *polynomial* identity remains true without restriction on the z's and x's. The necessity and sufficiency of $|\mathbf{R}(0)| = 0$ as a condition for $f(x)$, $g(x)$ to have a common zero is now obvious.

The above expression for $|\mathbf{R}(0)|$ can be obtained more easily by replacing the last column of $|\mathbf{R}(0)|$ by: last column + x_1 times $(m + n - 1)$th column + $\ldots + x_1^{\,m+n-1}$ times first column. This gives $x_1 - z_s$ $(s = 1, \ldots, m)$ as a factor of $|\mathbf{R}(0)|$, and the rest follows as before.

We could, of course, have proceeded from the original proof by noting that the x's and y's are continuous functions of the a's and b's; but it seems worthwhile, if only for aesthetic reasons, to put the argument in purely algebraic form.

Yet another method of proof, given in Bôcher, *Introduction to Higher Algebra*, is to consider the possibility of satisfying the identity $f(x)F(x) + g(x)G(x) = 1$ by polynomials F, G of degrees at most $n - 1$, $m - 1$ respectively.

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2932. The integral solution of a pair of equations

The simultaneous equations,

$$2x + y = 3z$$
$$2x^3 + y^3 = 3z^3$$

with x, y relatively prime, have the *unique* solution in integers

$$x = 4, \quad y = -5, \quad z = 1.$$

For, eliminating z, we find

$$5x^3 + 4y^3 = 3xy(2x + y);$$

thus 4 is divisible by x and 5 is divisible by y, and the only possible solutions are $x = \pm 1, \pm 2, \pm 4$; $y = \pm 1, \pm 5$. Trial shows that $x = 4$, $y = -5$ alone satisfies the equation, and the corresponding value of z is 1.

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2933. Beating the bookie

On opening the Times on Saturday 26th March, I observed the London odds for the 27 probable runners in the Grand National. They looked so inviting that I was led to consider them further and reached the following conclusion:

The odds on the various runners in a race are a, b, c, ... to 1. Then, by suitably backing every runner, so long as one runner (no matter which) completes the course, a profit will be made if

$$\Sigma \frac{1}{a+1} < 1.$$

Two preliminary results are needed:

1. It is better to bet on a straight win rather than "both ways."

Proof. To guarantee a win, every runner must be backed. Now backing "both ways" in effect increases the chances of backing a winner at the expense of lowering the overall odds.

But, if one backs every horse, one's chance of backing a winner can not be further increased.

Thus backing "both ways" would merely reduce the odds with no compensating advantages.

Therefore we need only consider bets for a straight win.

2. In order to get back a sum £S (this includes returned stake money as well as profit) on a runner winning with odds a to 1 it is necessary to stake £$S/(a+1)$.

Proof. A stake of £$S/(a+1)$ at a to 1 gives a profit of £$aS/(a+1)$ together with original stake £$S/(a+1)$. Total £S.

From these two results it follows that to guarantee a return of S,£ whatever the winner, the total stake must be $\Sigma S/(a+1)$, the

summation extending over all the odds. Thus there is certain unlimited profit if $\Sigma S/(a+1) < S$, i.e. if

$$\Sigma \frac{1}{a+1} < 1.$$

Incidentally, for this particular race, $\Sigma 1/(a+1) = 1{\cdot}0790$, a very close shave for the Bookies!

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2934. A certain type of theorem concerning two conics

In complex projective plane geometry, if two conics, S and S', have four distinct common points, A, B, C, D (and therefore four distinct common tangents a, b, c, d), there are theorems of the following type:

if the two conics have a special relation which involves pairing off the common points, then they have the same relation when the pairs are interchanged. For example if* the pole of AB for S' is the pole of CD for S, then the pole of CD for S' is the pole of AB for S. An example in which the relation involves only one pair at a time is: if the pole of AB for S' lies on S, then the pole of CD for S' lies on S. The dual theorems are true, of course. The relation may involve both common points and common tangents. For example, if a and b meet on AB, then c and d meet on CD.

All theorems of this type follow from the result that there is a linear transformation of the plane which leaves each conic invariant and which interchanges A and C and also B and D (and the common tangents similarly). If AC and BD meet at V, and l is the line joining the harmonic conjugates of V with respect to A, C and B, D, it is the harmonic homology with vertex V and axis l (that is, P is transformed into P', where PP' passes through V and P, P' are harmonically conjugate to V and the point where PP' meets l). In Euclidean geometry, when A and C are the circular points at infinity, so that both conics are circles, the transformation is the symmetry about the line joining the centres.

The theorems of the type mentioned sometimes have interesting special cases in Euclidean geometry when A and B are the circular points at infinity (so that again both conics are circles). The first theorem quoted gives the case of two circles cutting orthogonally. The second gives the result that if the centre of S' lies on S then the tangents to S' at C and D meet on S. The third gives the case of two circles of equal radii.

E. J. F. Primrose

* This appears, at first sight, to be a two-fold condition, but it is in fact a single condition: if the tangent to S' at A passes through the pole of CD for S, then so does the tangent to S' at B.

2935. To prove that every real number is an approximation to zero

If α is an approximation to a root of the equation $f(x) = 0$, then, by Newton's well-known formula, a better approximation is

$$\alpha - \frac{f(\alpha)}{f'(\alpha)}.$$

In particular, let

$$f(x) \equiv x,$$

so that

$$f'(x) = 1.$$

Thus

$$\frac{f(\alpha)}{f'(\alpha)} = \frac{\alpha}{1} = \alpha,$$

so that

$$\alpha - \frac{f(\alpha)}{f'(\alpha)} = \alpha - \alpha = 0.$$

If, then, any number α is taken as an approximation of the root of the equation

$$x = 0$$

(for example, if $\alpha = 1{,}000{,}000$), then *one single application* of Newton's formula gives the next approximation zero, which is absolutely correct.

Surely an approximation which yields the correct answer in one step is a good one.

E. A. Maxwell

2936. On note 2871

Professor G. N. Watson's generalization, in Note 2871, of Mr. H. G. Apsimon's Note 2754 is itself a particular case of a result due to R. Deltheil [1], which I rediscovered ten years ago [2]; namely that, if PQ is the distance between two points P and Q uniformly distributed in the interior of an m-dimensional hypersphere of radius a, then the mean value of PQ^n is

$$\frac{m\Gamma(m+1)\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2})(2a)^n}{(m+n)\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(m+\frac{1}{2}n+1)},$$

as given in a slightly different notation in equation (21) of reference 2. This reduces to Professor Watson's result when $m = 2$, upon using the duplication formula for the gamma function. Professor Watson assumes that $n > 0$; but the result holds for $n > -m$. If $n \leq -m$, the mean value of PQ^n is infinite. The object of the present note is less to allocate priorities, and more to point out that these results have practical applications: my own rediscovery of Deltheil's result was motivated by an enquiry into the damage

caused to plants by the presence of radioactive tracers in their fertilizers [3, 4, 5, 6]. The result in [3] was also a rediscovery of work by P. R. Halmos [7].

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[2] J. M. Hammersley "The distribution of distance in a hypersphere." *Ann. Math. Statist.* 21 (1950) 447–452.

[3] J. M. Hammersley "A theorem on multiple integrals." *Proc. Camb. Phil. Soc.* 47 (1951) 274–278.

[4] J. M. Hammersley "On a certain type of integral associated with circular cylinders." *Proc. Roy. Soc.* (A) 210 (1951) 98–110.

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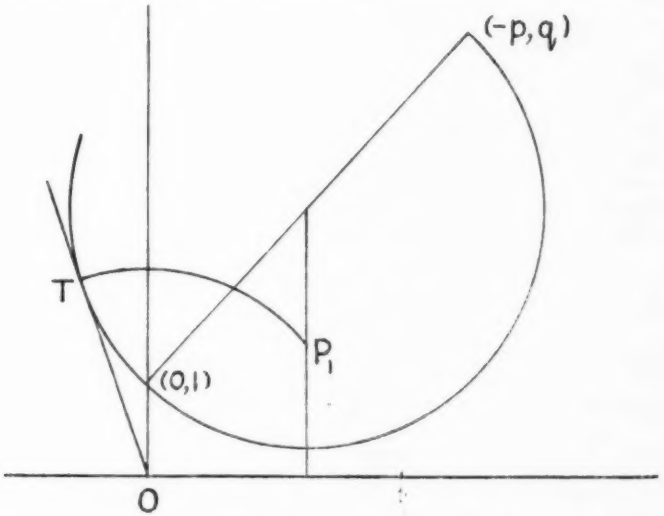
Institute of Statistics, Oxford J. M. HAMMERSLEY

59. The graphical solution of a quadratic equation with imaginary roots

Writing in a recent issue of *Praxis der Mathematik* (Cologne, June 1959) on the graphical solution of quadratic equations, K. Wigand gave the following constructions. To solve the equation

$$x^2 + px + q = 0$$

with real coefficients p, q we draw the circle on the line joining the points $(0, 1)$, $(-p, q)$ as diameter; the x-coordinates of the points



where this circle meets the x-axis are the roots of the given quadratic. If the circle does not meet the x-axis, then Wigand replaces the given equation by

$$x^2 + px + \tfrac{1}{2}p^2 - q = 0;$$

if the roots of this equation are $u \pm v$, then the roots of the original equation are $u \pm iv$.

Ingenious though Wigand's method is, I think it is preferable to have a construction in which the roots, whether real or imaginary, are found from the same circle. If the circle C on the line joining $(0, 1)$ to $(-p, q)$ as diameter does not meet the x-axis, we draw OT the tangent to the circle from the origin O, and find the points of intersection P_1, P_2 of the circle centre O, radius OT, with the line

through the centre of C, parallel to the y-axis. If the coordinates of P_1, P_2 are $(u, \pm v)$ then the roots of the given quadratic equation are $u \pm iv$.

For $v^2 = OP_1^2 - u^2 = OT^2 - (\frac{1}{2}p)^2 = q - (\frac{1}{2}p)^2$ and $u = \frac{1}{2}p$.

The pupil who later proceeds to study coaxial systems of circles will have the fun of recognising in this construction the theorem that circles orthogonal to a coaxial system pass through the limiting points (point circles) of the system, and that if the circles of the system do *not* meet the radical axis in real points, then the circles of the orthogonal system have real intersections with the line of centres.

R. L. GOODSTEIN

60. A general conic

The following simple result was only noticed by the writer recently, though it may perhaps already be well known to others. The equation below is worth recording, however, as it may have useful applications in problems where it is desirable to deal simultaneously in cartesian coordinates with hyperbola, parabola, ellipse and circle, while keeping the algebra as simple as possible.

The observation is merely that, when *any* proper conic is referred to axes which are the inward normal and tangent at a vertex*, its equation becomes

$$y^2 = 2Lx + (e^2 - 1)x^2,$$

where e is the eccentricity and L (chosen rather than l, to save any possible confusion in writing) is the semi latus rectum.

The centre is $\left(\frac{L}{1-e^2}, 0\right)$, the foci are $\left(\frac{L}{1 \pm e}, 0\right)$ and the directrices $x = -\frac{L}{e^2 \pm e}$ respectively. The condition for degeneracy is, of course, $L = 0$.

[Incidentally, by the application of Newton's formula, this form of the equation provides an interesting general result about the radius of curvature of *any* conic at its vertex.]

Hymers College, Hull ROGER F. WHEELER

61. The graph of $y = \log_a x$

The graph of $y = \log_a x$, as usually drawn, (Fig. 1), is only correct if $a > 1$. So also are statements such as: if $p < q$, then $\log_a q < \log_a p$.

* In the case of the ellipse, this term is used in the present context to denote an intersection of the conic with its *major* axis *only*.

Even though the bases 10 and e both have this property, that is no reason for ignoring the other case, (fig 2), which arises when $0 < a < 1$.

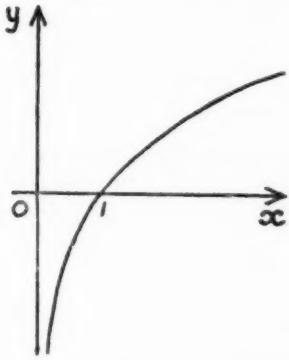


FIG. 1

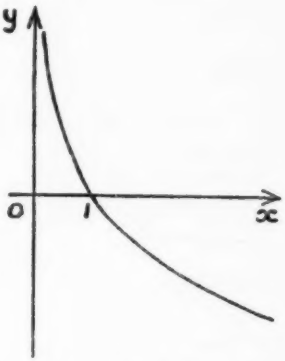


FIG. 2

There is, obviously, a similar dichotomy in the graphs of $y = a^x$, though this is less likely to be overlooked.

Hymers College, Hull ROGER F. WHEELER

62. A pair of equations

Mr. Hesselgreaves' suggested method in class room Note No. 1 is very interesting for young students.

Knowing the following fact, they may be encouraged to solve quadratic equations by factorisation.

$(a = 6, b = 5)$ is the only one pair of positive integers that a pair of equations

$$x^2 - ax + b = 0$$
$$x^2 - bx + a = 0$$

has two different roots of positive integers respectively.

Let two roots of (1) $\alpha, \beta(\alpha > \beta)$ and those of (2) $\gamma, \delta(\gamma > \delta)$, then we have

$$\alpha + \beta = a, \alpha\beta = b; \ \gamma + \delta = b, \gamma\delta = a.$$
$$\therefore a - b = \alpha + \beta - \alpha\beta = 1 - (\alpha - 1)(\beta - 1)$$
$$= \gamma\delta - (\gamma + \delta) = (\gamma - 1)(\delta - 1) - 1.$$

Suppose $a > b$, then we have

$$(\alpha - 1)(\beta - 1) < 1.$$

This implies

$$\beta = 1 \quad \text{and} \quad a - b = 1.$$

Hence

$$(\gamma - 1)(\delta - 1) = 2 \quad \text{and} \quad \gamma = 3, \delta = 2.$$

Thus we obtain

$$a = 6, \quad b = 5$$

and

$$x^2 - 6x + 5 = (x - 5)(x - 1)$$
$$x^2 - 5x + 6 = (x - 3)(x - 2).$$

Kyushu University, Japan T. NAKAZAWA

63. On Notes 2823 and 2879

An alternative method, useful from a teaching point of view, of dealing with differential equations, whose auxiliary equation has equal roots, is as follows.

The equation

$$\ddot{y} + 2\alpha\dot{y} + (\alpha^2 + \beta^2)y = 0 \tag{1}$$

has the solution

$$y = e^{-\alpha t}(A \cos \beta t + B \sin \beta t) \tag{2}$$

If, with (1), we have the conditions that, when

$$t = 0, \quad y = Y \quad \text{and} \quad \dot{y} = V,$$

from (2) we have

$$y = e^{-\alpha t}(Y \cos \beta t + \frac{V + \alpha Y}{\beta} \sin \beta t) \tag{3}$$

If we now let $\beta \to 0$ we have the case of the equation

$$\ddot{y} + 2\alpha\dot{y} + \alpha^2 y = 0 \tag{4}$$

with the solution

$$y = \lim_{\beta \to 0} e^{-\alpha t}(Y \cos \beta t + \frac{V + \alpha Y}{\beta} \sin \beta t)$$
$$= e^{-\alpha t}\{Y + (V + \alpha Y)t\}$$

Comparison of the coefficients of $\sin \beta t$ in (2) and (3) then suggests the following method of obtaining the formula for the general solution of (4).

$$y = \lim_{\beta \to 0} e^{-\alpha t}[A \cos \beta t + B \sin \beta t]$$
$$= \lim_{\beta \to 0} e^{-\alpha t}\left[A \cos \beta t + \frac{B'}{\beta} \sin \beta t\right]$$
$$= e^{-\alpha t}[A + B't]$$

If the case of equal roots can be postponed until the students have had some practical applications, then, since equation (1) is that for the free oscillations of a system with damping coefficient, α and damped natural frequency $\beta/2\pi$, discussion of the case of critical damping ($\beta = 0$) leads quite naturally to the examination of the form of (2) as $\beta \to 0$, so that the method of solution arises in an unforced manner.

Rutherford College of Technology J. PRITCHETT

64. The envelope E of the normals to a curve C is the locus of the centre of curvature.

Let X, Y be the co-ordinates of any point on C, and consider them as functions of s. s, ψ are the usual intrinsic co-ordinates of points on C.

The equation of the normal at (X, Y) is

$$(x - X)\cos\psi + (y - Y)\sin\psi = 0 \qquad (1)$$

Differentiate w.r.t. the parameter s

$$-\frac{d\psi}{ds}\cos\psi - (x - X)\sin\psi\frac{d\psi}{ds} - \frac{dY}{ds}\sin\psi + (y - Y)\cos\psi\frac{d\psi}{ds} = 0$$

$$\text{i.e.} \quad -(x - X)\sin\psi + (y - Y)\cos\psi = \frac{ds}{d\psi} \qquad (2)$$

since

$$\frac{dX}{ds} = \cos\psi \text{ and } \frac{dY}{ds} = \sin\psi.$$

The solutions for x and y of 1 and 2 are the co-ordinates of the point of contact of the normal and E. Squaring and adding 1 and 2

$$(x - X)^2 + (y - Y)^2 = \left(\frac{ds}{d\psi}\right)^2$$

i.e. the distance along the normal from the intersection with C to the point of contact with E is the reciprocal of the curvature.

W. W. O. SLESSENGER

65. Some varied approaches to elementary functions

Graphs in Sixth Form work

Each of the Mathematical Association's Reports[1 and 2] on the teaching of Algebra has an inspiring section on 'Graphs', and the subject receives some mention in the G. C. E. Advanced syllabuses of certain Boards. In spite of this, however, it is my experience that students who enter Training College after successful Sixth Form work in Mathematics have very meagre knowledge of the graphs of functions. Perhaps other Sixth Formers too are being denied the chance of creative work in this rich field of mathematics. I should like to make a plea for the inclusion in school of such work on 'Graphs' as the Reports recommend ; and I venture to add some related topics which I have found of interest to Training College students, and which might be of use for occasional work with Sixth Form pupils.

Inverse functions

These are usually introduced in the Calculus course, after work on the circular functions. In many text-books, and in the Association's 'Calculus' Report,[3] the only inverse functions treated are those of the circular, exponential and hyperbolic functions, and the graphs of a function and its inverse are nowhere compared. This seems a pity—surely the concept of an inverse is interesting and significant enough to warrant a little more time spent on it? The treatment I have in mind could follow this sequence:—

1. The first inverse functions met can be x^2 and $\sqrt{x}$; their graphs can be sketched on the same axes, with the consequent discovery that each is a reflection of the other in the line $y = x$. The printed tables of these functions can also be discussed and compared.

2. The table of sin x (or tan x) can now be considered, where x is in degrees. Printed tables used in school generally contain no 'anti-sine' table, but it is an easy task to construct one, making sufficient entries to give it some use in solving problems of numerical trigonometry.

3. The exponential and logarithmic functions can have varying treatments, according to the stage at which they are first met. In an Appendix of the 'Algebra in Schools' Report[1] is suggested a sequence of work on the graph of a^x, presumably for younger pupils than Sixth Formers; the 'Calculus' Report[3] gives two approaches to the subject. As regards graphical work, the inverse nature of these two functions is best seen by plotting the graphs in pairs on the same axes; e^x and $\log_e x$ are of course the 'basic' functions, but if 10^x and $\log_{10} x$ are also plotted together the students realise the true nature of the Antilogarithm table.

Such activities as these, in my experience, give students a deeper awareness of several of the fundamental concepts. I add also a sequence of work on functional scales, which could conveniently follow the above, and a further paragraph on a topic to which it is relevant.

Functional scales

Nunn's 'Exercises in Algebra' has an exciting chapter in which the student is led to construct a logarithmic scale by use of the graph of a^x. The Appendix of the 'Algebra' Report,[1] referred to above, shows how the study of this graph throws light on the change of base of logarithms, but stops short of the construction of the logarithmic scale. I have found that students welcome and extend the method given in Nunn's book, which is as follows:—Using any suitable curve a^x, make a set of graduations on a new scale parallel to the x-axis, which record the heights of the ordinates immediately above them; that is, choose a suitable ordinate for '1', and record its position on

the new scale, then find the position of the ordinate whose height is double this and label it '2', and so on.

As Nunn says, the logarithmic scale thus built up can immediately be used to make a simple slide-rule. Better slide-rules can then be made by using linear-logarithmic graph paper.

It is natural now to try the method for other functions, and to prove that the graph of any function may be used to obtain, by drawing, the scale of its inverse. Thus, the scale of square roots can be constructed from the scale of squares, and vice versa. The scale of reciprocals, constructed from one branch of the hyperbola $y = k/x$, is particularly interesting. Occasionally, students may wish to make a slide rule of 'squares' (which has some use for obtaining the third side of a right-angled triangle), or a slide rule of reciprocals (a pleasant luxury for finding the third variable in the formula $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, if two are already known).

Functional scales can of course be constructed directly. Their special uses are interesting. For instance, the exponential scale is used in statistical experiments when flow-in at random is simulated; and the scale of chords, which can usefully be compared with the scale of sines, appears on the wooden 6-inch rulers used by draughtsmen.

The making of scales brings out the contrast between single-valued and multivalued functions. In the case of the latter (for instance, the square root function or the inverse sine function) there are two or more points on the scale with the same label. Here it is clear that the function is not uniquely defined unless a restriction is made on the range of values used.

The testing of suspected laws of connection between two variables

Finally, the work done above on scales can be put to further account when the students meet the type of problem in which the relation between two variables has to be determined from numerical data. The Report 'Algebra in Sixth Forms'[2] has a paragraph on this type of problem, and says that it is good for the class to have the experience of using logarithmic graph paper. Here the reference is to 'log-log' paper, in connection with the case in which a relation of the form $y = a \,.\, x^n$ is expected; it could apply also to 'linear-log' paper, used for the formula $y = a \,.\, b^x$. Students who have experimented with functional scales, as above, take readily to the technique of investigating relations by selecting the right functions of the given variables and are already familiar with logarithmic graph paper.

[1] The Teaching of Algebra in Schools (1934, revised 1956).
[2] The Teaching of Algebra in Sixth Forms (1957).
[3] The Teaching of Calculus in Schools (1951).

City Training College, Sheffield M. J. MEETHAM

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

May I draw the attention of your readers to a matter of mutual interest to teachers and training college lecturers?

The increase in the length of the course of training for non-graduate teachers from two to three years is necessitating an increase in staffs of training colleges and this is likely still further to denude secondary schools of their mathematics teachers and so aggravate an already serious situation. The committee of the Mathematics Section of the Association of Teachers in Colleges and Departments of Education has given some thought to the problem and some colleges have been able to solve their staffing difficulties by appointing retired teachers or lecturers on a part-time basis. It is felt that more use could be made of this potential source of supply if there existed some central coordinating body for the collecting and collating of information. The Mathematics Section is willing to act in this capacity for the time being and any teacher or lecturer who is contemplating retirement and who would be willing to undertake part time work in a training college is invited to communicate with the undersigned at Redland College, The Promenade, Bristol, 8.

Yours etc., KATHLEEN SOWDEN

To the Editor of the *Mathematical Gazette*

DEAR SIR,

May I add a note to the interesting article by Mr. R. F. Wheeler on Force, Power and Gravitational Units printed in the December 1959 issue of the Mathematical Gazette. The title is

Pound Weight and Pound Mass

The confusion between these two terms arises partly from history and partly from the changes that have occurred in thought.

In Kaye and Laby's Tables of Physical and Chemical Constants (Longmans Green and Co. 1911) First edition under the heading of British Imperial Standards on p. 4 we find this statement "According to the Weights and Measures Act, 1878 the pound is the weight in vacuo of a platinum cylinder called the imperial standard pound." Note that the pound is *a weight* (or force) and *not a mass*.

The eleventh edition (1956) of the same tables on p. 3 changes the above to "The Imperial Standard Pound, defined by the Weights and Measures Act, 1878, is a cylinder of platinum of diameter slightly less than its height etc. The Standard Pound defines the avoirdupois pound". There is no mention here of weight or mass, the pound *is* the cylinder of platinum, although the related paragraphs are under a central heading MASS.

In 1928 the British National Physical Laboratory published a report on The Units and Standards of Measurement employed at the N.P.L. It quoted (p. 23) the same Act and says "The Imperial Standard Pound is defined as follows: The imperial standard for determining the weight

of the imperial standard pound *is of* platinum the form being that of a cylinder etc.

In March 1946 the N.P.L. published another report "A Discussion on Units and Standards" which was reprinted from Proc. Roy. Soc, A Vol. 186, 1946. Under the heading "The Standards of Mass" it says "The fundamental standard in the British system of units is the Imperial Standard Pound (Weights and Measures Act, 1878). This was constructed in 1844 in the form of a cylindrical piece of platinum.... It is to be noted that this standard is the avoirdupois pound containing 7000 gr." Again no mention of weight, but one notices a very gradual introduction of mass through the title of the section.

The section goes on to say "In the metric system it is of interest to refer to the original conception underlying the definition of the unit of mass. The kilogram was originally defined by reference to a "natural" standard, i.e. the mass of the cubic decimetre of water. The material representation of this standard was a simple cylindrical piece of platinum. ...

Here the word mass is employed with no mention of weight, and it would be of interest to know which of these two terms was employed in those days in France when the Metric system was invented.

In 1955 the United States Department of Commerce (National Bureau of Standards, Miscellaneous Publications 214) published a report on Units of Weight and Measure. Note the use of the singular. Here both the kilogram and the pound are defined as units of mass, indeed the pound is defined in terms of the Kilogram. Weight is not mentioned except in the title of the report.

In 1956 the Government of Canada passed an Act respecting Weights and Measures (15, George VI) and says on p. 194 of the printed report "The pound is the only unit or standard measure of weight from which all other Canadian weights and measure having reference to weight are derived". So it looks as if the word weight will remain in popular usage while mass will be oftener used in scientific writings. It would be interesting to know if the term "unit of mass" has legal sanction in England.

If, as physicists say, the pound is the unit of mass in the British system the unit of force must be the weight of one g'th of the weight of this pound i.e. the weight of about half an ounce. It is called the *poundal*.

If, as engineers say, the pound is the unit of force in the British system the unit of mass must be the mass of a lump of matter whose mass is g-times the mass of this particular pound. It is called the *slug*.

The engineer who deals with statical problems only is safe with his definition of the pound as a force, e.g. in the use of the abbreviation "psi"for a pressure. But when the engineer is engaged in a dynamical problem he has to be careful to insert the g where necessary. If a mass M pounds has a velocity V ft per second the physicist says it has Kinetic energy $\frac{1}{2}MV^2$ foot poundals. The engineer referring to a weight of W pounds having a velocity V ft per second has to say the Kinetic Energy $= \frac{1}{2}\frac{W}{g}V^2$ foot pounds. The physicist would measure pressure

in *pounds-weight* per square inch, or more likely in dynes cm^{-2}. Many of us remember our school boy days when, after working out a problem in Dynamics, we looked up the answers to see whether we had to divide by g to make our result agree with the answer.

Yours etc., JOHN SATTERLY

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Mr. Whitfield in his review of my book on "Three-Dimensional Dynamics", which appeared in Vol. XLIII, No. 346, of the Mathematical Gazette, after congratulating me on giving a correct proof of the variational principles in impulse theory (Kelvin's and Robin's Theorems) goes on to say that my statement that "Bertrand's Theorem involves no stationary property" is false. This is a statement which I think needs clarifying.

Bertrand's Theorem states that the kinetic energy of any free system when set in motion by a set of impulses is greater than that of the same system when subject to frictionless constraints and set in motion by the same impulses. If frictionless constraints can be imposed on a system, the constraints being such that they can be so continuously varied that the resulting motion differs by as little as one pleases from the actual motion of the free system, then Bertrand's Theorem can certainly be associated with a stationary property, since in the result

$$\tfrac{1}{2}\Sigma m(\mathbf{v}^2 \,.\, \mathbf{v}^2) = \tfrac{1}{2}\Sigma m(\mathbf{v}_2' \,.\, \mathbf{v}_2') - \tfrac{1}{2}\Sigma m(\mathbf{v}_2 - \mathbf{v}_2') \,.\, (\mathbf{v}_2 - \mathbf{v}_2'),$$

where $\mathbf{v}_2$ corresponds to the free system and $\mathbf{v}_2'$ to the constrained system, we can replace $\mathbf{v}_2'$ by $\mathbf{v}_2 + \delta\mathbf{v}_2$ giving

$$\Sigma m(\mathbf{v}_2 \,.\, \delta\mathbf{v}_2) = 0,$$

i.e.

$$\delta\Sigma m(\mathbf{v}_2 \,.\, \mathbf{v}_2) = 0,$$

so that the actual motion corresponds to a stationary value of the kinetic energy. The actual motion in this case corresponds to the constrained motion which has the maximum kinetic energy. Thus, for instance, suppose we consider a uniform rod AB of mass M and length $2a$ set in motion by an impulse J applied at A at right-angles to AB. Let the motion be defined in terms of $\mathbf{v}$, the velocity of G the centre of mass, together with $\boldsymbol{\omega}$, the angular velocity of the rod. The direction of $\mathbf{v}$ will clearly be at right-angles to AB in the direction of J. Hence, taking $\boldsymbol{\omega}$ to have the appropriate direction, the equations to determine the motion are

$$Mv = J, \qquad I\omega = aJ,$$

I being the moment of inertia of the rod with respect to an axis through G perpendicular to the rod. We thus have

$$v = J/M, \quad \omega = aJ/I = 3J/Ma.$$

Now we can clearly apply a frictionless constraint to the system by fixing a point of the rod by means of a smooth pin, and the motion as

given above can be obtained by making the kinetic energy of this constrained motion a maximum for variations in the position of the pin. The constrained motion which corresponds to the motion of the free rod is that with the pin placed at the instantaneous centre. The same result applies, of course, whatever the point at which the impulse is applied. If, however, the impulse is applied at a point distant from the centre $>a/3$, then the instantaneous centre is a point of the rod, but if the distance of the point from the centre is $<a/3$, then the instantaneous centre of the motion is a point outside the rod, which makes the nature of the constraint applied in this case somewhat unreal. To say that a stationary property can be associated with Bertrand's Theorem as a general result, then one should be able to say that, no matter what the system, one can always impose *real* frictionless constraints on the system of such a nature that they may be so varied that the resulting constrained motion differs by as little as one pleases from the motion of the free system. Clearly this is not always possible. Thus, while Bertrand's Theorem can be associated with a stationary property in particular cases, one cannot say that it can be so associated as a general principle. This is the point of the statement in my book that Bertrand's Theorem involves no stationary property "since it does not follow that frictionless constraints can be imposed on the system in such a way that the $\mathbf{v}_2$'s and the $\mathbf{v}_2$''s corresponding to the different systems differ from one another by infinitesimal amounts". I feel, however, that this statement does not in itself make the position completely clear, and I am pleased to have this opportunity, arising out of the review of my book by Mr. Whitfield and correspondence with him, to clarify more exactly the variational implications of Bertrand's Theorem.

Yours etc., C. E. EASTHOPE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

QUERY:—

A common form of crossword frame consists of a square, divided into rows and columns each containing 15 small squares, some of which are blacked out to form a centrally symmetrical pattern. Assuming that

(i) No row or column is completely blacked out;
(ii) No 'word' consists of more than 13 or less than 3 letters;
(iii) At least one letter of every 'word' is shared by a 'word' in the other direction;

is it possible to determine just how many frames can be constructed?

Yours etc., B. A. SWINDEN

SAMUEL PEPYS AND JOHN WALLIS

1952. *From Samuel Pepys to Sir Godfrey Kneller, March, 26, 1702.*

I have long, with great pleasure, determined, and no less frequently declared it to my friend Dr. Charlett, upon providing as far as I could by your hand, towards immortalizing the memory of the person, for

his fame can never die, of that great man, and my most honoured friend, Dr. Wallis, to be lodged as an humble present of mine, though a Cambridge man, to my dear Aunt, the University of Oxford.

(The portrait was duly executed by Kneller and presented by Pepys to the University. It now hangs on the staircase of the Bodleian Library.)

Wallis on the Tuning of Organs

Dr. William Holder, a former Sub-Dean of the Chapel Royal and one of the earliest Fellows of the Royal Society, gave in his book of 1694 a general outline of Meantone tuning, which involved flattened 5ths, perfect major 3rds, and (as was well understood) only a limited range of available keys. Four years later the Oxford mathematician Dr. John Wallis (to whom Pepys had sent the organ-builder Harris for advice) gave his opinion that since the pitch of organ pipes could not be instantaneously corrected 'as the Voice may be by the guidance of the Ear', organists who tuned all their semitones equal were doing the best they could, short of 'multiplying intermediate Pipes'. This was a vote for EQUAL TEMPERAMENT. (From John Wilson: Roger North on Music).—[Per E. H. Lockwood, Felsted School, Essex.]

The Teaching of Mathematics—1. The development of number concepts in young children, by J. B. Biggs; **2.** Attitudes to Arithmetic —number anxiety, by J. B. Biggs; **3.** The growth of mathematical concepts in children through experience, by Z. P. Dienes. *Educational Research*, Feb. 1959, p. 17; June 1959, p. 6; Nov. 1959, p. 9.

Practising teachers are apt to divide articles on the theory of education into two classes; (a) accounts of psychological and statistical research in which the maximum labour has been expended in establishing a few obvious results, as, for example, that other things being equal, the lazy child gets fewer marks; and (b) speculative opinions based on the minimum of facts and bearing no relation to the real classroom situation in which 45 pupils of widely varying ability have to be prepared for an examination in three months' time. Such readers would immediately classify Dr. Biggs' articles as of type (a) and Dr. Dienes' as of type (b). This would be unjust, for while Dr. Biggs' two papers are fully documented, and some of the investigations referred to yielded little of positive interest, yet his main conclusions are interesting, valuable, well-authenticated and expressed with scholarly caution. Dr. Dienes' third paper is admittedly more speculative and the editors themselves admit that its experimental basis is less than they usually demand; nevertheless it is essentially practical in outlook and the continuation of the experiment it describes is very desirable, since it promises to get interesting results.

Dr. Biggs begins with an exposition of Prof. Piaget's work on the *Psychology of Intelligence*. In this he distinguishes three levels of general intellectual growth (figures in brackets denote approximate ages): (i) *non-operational* (0–2) in which concepts do not exist, but only objects; a ball not seen ceases to be; (ii) *pre-operational* in two stages: (a) transductive (2–4) involving memory, but no distinction between general and particular; and (b) intuitive (4–7) based on crude perception; what *looks* bigger *is* bigger; (iii) *operational*, again in two stages: (a) concrete (7–11), based on reversibility of thought, e.g. the passage from $2 + 6 = 8$ to $8 - 6 = 2$, and conservation of concept, which permits generalization; (b) formal (11 +) in which abstraction becomes possible. Within this general framework, there is at any stage a recapitulation on a smaller scale of the three main stages when any new concept is introduced: stage one when the concept does not exist at all—in the case of a number concept this will be because the child is still at the stage of crude perception (iib); stage two, when a pre-concept is formed but its constancy is not recognized; stage three, when the concept becomes fully operational, and indeed appears 'obvious'. The exposition is not very clear at this point, but this is the meaning which the reviewer believes to be intended. It is now clear that passage to stage three, or even stage two, in the appreciation of any particular concept, is impossible unless the appropriate stage of general intelligence has been reached.

Dr. Biggs treats this theory with caution, and analyses a great many actual examples in the light of it. In particular he criticizes the rigid

age-ranges given by Piaget and gives evidence to show that they are far more variable, flexible, and modifiable by appropriate teaching. The reviewer finds cause to suspect some of the evidence on the grounds that it appears to have been assumed that a child learns *only* what he is taught in school! By the mercy of providence this is not the case. Children's number perception is conditioned by a wide variety of everyday experiences, and not only by the blocks they play with in the infants' school. But certain implications are convincing. If a child cannot *match* and *seriate* (which is not the same as mere counting, which may be a recital of words as meaningless as "eeny meeny miney mo") then he is not *ready* to add; if he is forced to do so by drilling he is likely to do very poorly, and to acquire a distaste for arithmetic which will be difficult to eradicate.

Biggs then discusses teaching methods, and in particular the controversy as to the merits of "drill" and "meaning" methods. The conclusions here are rather obvious. (i) 'Drill' is better for speed at a specialized task, but 'meaning' is better if skill has to be transferred to new situations. (ii) 'Drill' may produce more immediate results, but 'meaning' produces better retention of skill. (iii) 'Meaning' is essential for difficult concepts, but if less intelligent children *must* master them, 'drill' must be used. (iv) Children *prefer* to be taught meaningfully, although many find pleasure in mechanical work.

Biggs does not regard the use of concrete objects in number work as a panacea for all ills; indeed it may actually hinder concept formation. This point is taken up fully by Dienes. He emphasizes the need for widely varying materials to help the child abstract the constant concept. 4 + 2 is not a matter of four sticks and two sticks, or four pennies and two pennies, or four blocks and two blocks, or even 1111 and 11, but what is common to all these; the child needs a wide variety to help him seize on the common element. Dr. Dienes sets himself in this article to elucidate his view of what Dr. Biggs means by "meaningful" teaching; his answer is the "structured game." (The word "game" is extended to include experiments of all kinds). This is an activity designed to establish a new concept; e.g. hanging weights on a beam at different distances from the pivot in order to discover conditions of balance, so leading to the concept of "moment". This, when grasped, is confirmed by "practice games", which would presumably include routine paper work. Dr. Dienes applies these principles to two cases; the use of similar figures of various shapes to establish the concept of a "square number"—how many identical triangles must be used to build a triangle three times as much each way?—and even to establish such results as $(a + b)^2 = a^2 + 2ab + b^2$; and the use of "multibase blocks" to establish positional notation, *not merely to base ten, but to a wide variety of bases.* He claims that this is not difficult for children—though it may confuse an adult with preconceptions! But when he goes on to use similar material to establish trinomial factors and logarithms, we must ask (what unfortunately is not stated) what age of child is envisaged!

I have so far omitted reference to Dr. Biggs' second article. This is a self-contained account of research into the reasons why boys and (more often) girls dislike arithmetic. Certain specific results emerge. If an

otherwise intelligent child is very poor at arithmetic, there is strong reason to suspect emotional disturbance. If such a child is accused of laziness and driven hard, a serious 'number-anxiety' may be fixed. Dislike of arithmetic may be due to lack of understanding or lack of interest; not usually to dislike of the teacher, but often to the teacher's own dislike of the subject. Such attitudes often arise early in the infants' school. The responsibility of the girl from the training college who "could never do very well with her sums" is therefore very great. If all teachers, in short, could be persuaded to read these articles, it would make us think; and we need to be made to think. We shall often disagree, and disagree violently with some of the conclusions; indeed the writers themselves disagree with other pundits and advocates of special methods and apparatus. But at least we may 'think it possible that we are wrong'; that what we have always done is not necessarily the right thing to have done; and so catch something of the humble spirit of enquiry which animates the writers of these three most interesting papers.

H. M. CUNDY

Konstruktive Abbildungsgeometrie. By MAX JEGER. Pp. 79. 1959. (Verlag Räber & Cie., Luzern.)

As long ago as 1868 Felix Klein advocated that, in teaching geometry in secondary schools, Euclid's methods should be replaced by newer methods based on the idea that any geometry is the study of those properties which are invariant for a certain group of transformations. Today, although Klein's ideas have been adopted in much of the geometrical teaching in universities, they have still not been generally adopted in schools, with the result that many students find the gap between school and university geometry difficult to bridge. This book has been written in an attempt to introduce this modern outlook into secondary schools in Switzerland.

Various types of transformation of the plane are studied, and the author gives several construction problems which can be solved by means of transformations of the appropriate type. He considers those transformations which form groups, and shows that some of these groups are subgroups of others.

I found this book exciting to read. It will be interesting to see what effect it has on the teaching of geometry in Switzerland. If it is successful, it will have two beneficial effects. Firstly, it will introduce to future university students two of the main ideas of modern mathematics, transformations and groups. Secondly, it will give to those who are not going to a university a powerful method for solving practical geometrical problems.

University of Leicester E. J. F. PRIMROSE

A Preparatory Algebra. By H. L. M. HOUSEHOLD. pp. 128. 5/6. 1959. (University of London Press Ltd)

This small book is designed to provide an introduction to algebra for pupils in Preparatory Schools or the lower forms of Grammar Schools. The sequence of ideas is sound, emphasis being laid on the 'shorthand

of arithmetic'. The numerical treatment is very thorough, but there is little cross-reference to related ideas in geometry. There are two chapters on graphical work, but even here illustrations are few. Each chapter has abundant exercises, and each set has been carefully graded. The type face is particularly clear, and great care has been taken in the lay-out of the worked examples. The book does what it sets out to do and thoroughly covers all the algebra that candidates for the Common Entrance Examination are required to know, and would equally well serve as a textbook for pupils in the first two years in a Grammar school.

J. W. COLLEY

General Mathematics. By J. B. CHANNON and A. MCLEISH SMITH. Vol. III (without answers) pp. xiv + 351 8/6 1958; Vol. IV (with answers), pp. xv + 388. 9/6. 1959. (Longmans Green & Co.)

These two volumes complete the series begun in 1955 and designed to cover the mathematics needed for the various syllabuses of the ordinary level examinations for G.C.E. The four books do this most adequately, the last of them providing a good introduction to 6th form studies. Books III and IV continue the same pattern followed in Books I and II with the same thoroughness of treatment. Topics are taken in reasonably sized chapters (24 in Book III and 27 in Book IV) dispersed throughout the book, so as to provide a varied but logical sequence. The authors have been at pains to provide useful ancillary exercises where these are appropriate; for example, on the handling of sums and differences of decimal numbers in the chapter on quadratic equations. Book III contains good chapters on everyday arithmetic, and on plans and elevations as well as the usual topics found in most middle school books. Book IV winds up the course with a number of diverse topics required by some, but not all, the examining bodies. The concluding chapters on the calculus are sound and cover the essential needs of the pupil at this stage. The complete set will, I think, meet with general approval in schools and deserve to be widely used.

J. W. COLLEY

Teach Yourself Arithmetic. By L. C. PASCOE. pp. xv + 232. 7/6. 1958. (English Universities Press Ltd.)

There is something a little ironical in asking a teacher of mathematics to review a book which sets out to make him redundant! But this book is not intended for the unwilling schoolboy, but rather for the more mature student who has a desire to study on his own. To this end more space is given to verbal explanation and diagrams than is usual in an elementary textbook. The author assumes that the student has no previous knowledge and begins in chapter two with elementary processes. (Chapter one is a rather sophisticated introduction, which a student who was really uninformed mathematically might find confusing.) The sequence from chapter two to chapter sixteen is that followed in many school texts and covers the requirements of most examinations in

arithmetic as well as the 'everyday' topics of interest, rates and taxes, insurance, and investment. The final chapter on computing machines and the binary scale is an interesting addition to the course. The self-taught pupil who gets through to this chapter and is able to appreciate it can certainly congratulate himself on a remarkable achievement. And the author, too, is to be congratulated for having made such a thing possible.

J. W. COLLEY

Le matériel pour l'enseignement des mathématiques. By G. GATTEGNO, W. SERVAIS, E. CASTELNUOVO, J. L. NICOLET, T. J. FLETCHER, L. MOTARD, L. CAMPEDELLI, A. BIGUENET, J. W. PESKETT, P. PUIG ADAM.

This symposium contains some of the advanced thinking about the psychology and the teaching of mathematics of the past few years. The greater part of the work is devoted to Geometry; fixed as well as movable models are examined and a great deal is said about instruction by mathematical film. The dynamic nature of learning, it is claimed, makes static models less suitable and nearly all contributors accordingly come down in favour of moving or movable aids. The essentially individual nature of the dynamics of learning is stressed by L. Campedelli, who advocates: "personal and direct experience, without any kind of mediation, apart from some discreet urging by the teacher in the direction required." E. Castelnuovo criticises the use of models if they are given to children ready made with which they can *do* nothing, realising the essential part played by individual experience during the learning process. She strikes a very sound note when she reminds us that already for Pestalozzi intuition meant construction; this is in line with modern research in cognition. Construction in the learning of mathematics is probably the most important single factor operative in such learning, and equally probably the one that has been almost universally neglected. If the present volume does no more than draw attention to the need to emphasise construction (and so ipso facto experience) in mathematics learning, then it will have been well worth producing.

Far the most important chapter is the one contributed by W. Servais on the relationship between the concrete and the abstract. The dynamic relationship between these two aspects of our apprehension of our environment is very well brought out. The fact that the same thing can be abstract at one moment of time and concrete at another, depending on the stage of thinking that has been reached, cannot be stated too often. The function of language and of symbols in general as vehicles aiding the "concretisation" of the abstract, is put very forcibly and lucidly by the author. The ideas expounded in the chapter are entirely in line with the direction of modern research in the psychology of thinking. Although it is a difficult chapter for the non-psychologist, it should be read by all teachers of mathematics at whatever level.

Upon looking over the volume one cannot help being struck by the preponderance of geometrical problems considered. It is well that we

should be encouraged to use material of a more concrete kind in the learning of Geometry, even though in this branch of Mathematics concrete representations are not lacking by the nature of the subject. It seems curious, however, that in Arithmetic, Algebra, Theory of Functions, of Limits and in similar branches, the practising teacher is not offered any ways of putting the modern principle of dynamism into practice in the classroom. One cannot help feeling that perhaps E. Castelnuovo's warning about the use of material might apply to the mathematical film or to any other form of "demonstration". The discovery of mathematical truths is telescoped into a few minutes of film or explanation of a model; this may not give enough room for the functioning of the construction aspect of the thinking process. It is hoped therefore that some further thinking will be directed to the didactic problems of *all* branches of mathematics as well as to the practical problem of how to enable individual children to make as many of their discoveries as possible by constructing the intuitions they gain from their personal and direct experience.

Z. P. DIENES

Mathematical Ideas. Their nature and use. By JAGJIT SINGH. Pp. 312. 35s. 1959. (Hutchinson)

To expound mathematical ideas to the layman while avoiding the complexities of mere technique is an important but difficult task, for the best mathematics is a fusion of ideas and technique. The author has not taken the easy way of ignoring modern and difficult concepts, and on the whole he has been remarkably successful. The chapters on groups and on probability are particularly good: the careful and conscientious reader should obtain a genuine insight into these theories, and a stimulus to clear and accurate thought on the topics. Some of the elementary work is less successful: formal algebra and calculus are swallowed in gulps, a process which may very well lead to mental indigestion.

The author has a thesis, that mathematics springs from social and economic demands and is not a "free creation" of the human mind, that it perpetually degenerates into pedantry and is then restored and revivified by the impact of physical problems. This, to me, is as lopsided as the thesis that mathematics is a purely abstract exercise of the intelligence, but one would not quarrel with the author were it not that to defend his point of view he sometimes resorts to shifts which are disingenuous. For example, in the chapter on vectors, we are told that Maxwell's electromagnetic theory demanded an extension from number to number associated with direction. Vector algebra is then described and at the end of the chapter we learn that this calculus was devised by Hamilton and Grassmann. From this arrangement, the layman will almost certainly get the impression that the work of Hamilton and Grassmann was a response to the demands of Maxwell's theory, since the corrective dates, 1844 for Hamilton and Grassmann, 1873 for Maxwell, are nowhere given. It is a pity that the author takes this unbalanced attitude, since he could easily have described the continuous

give and take between the intellectual curiosity of the mathematician and the practical demands of the physical world. Jagjit Singh's case is not unanswerable: mathematical history shows us that Fourier did not silence Jacobi, nor did Airy annihilate Cayley.

T. A. A. B.

Fundamental principles of mathematics. By J. T. Moore. Pp. xv, 630. $7. 1960. (Rinehart, New York)

A course in mathematical analysis. I. By N. B. Haaser, J. P. LaSalle, and J. A. Sullivan. Pp. xiv, 688, xxxi. $8.50. 1959. (Ginn)

These two books, for the American freshman year, have a common aim, to bridge the gap between "school" mathematics and "modern" mathematics. The first-year undergraduate frequently has trouble in coping with a two-fold difficulty, more advanced topics on the one hand, more severely logical methods on the other; ideally these should be tackled one at a time, by surveying the "school" mathematics again under a logical searchlight, but the practical difficulty is to find the time for this. It is possible that in this country we might do more in the last year at school, if we cut out of the A level course some of the unnecessarily complicated elaborations of elementary techniques—laborious exercises on the trigonometry of the triangle, for instance.

Mr. Moore's book is the more elementary of the two; he begins with new concepts, such as set theory and the real number system, and deals with them at slow pace; he then passes in review almost the whole field of school mathematics, algebra, trigonometry, coordinate geometry, the beginnings of the calculus, probability and statistics. New concepts are introduced smoothly, and old ideas are clarified and tightened up. A novelty is the dual use of radian-circular and degree-circular functions; it is hoped that this will eliminate the confusion which causes the area under a curve to be found in degrees. Care is taken over proprieties of language, as when the measure of a physical quantity is in question, and where proofs are not given the omission is noted, not concealed. In one or two places, the author falls below his own standards; thus, in dealing with the solution of equations, and particularly when extraneous roots are introduced by squaring, he quite fails to make the important logical distinction between reversible and irreversible steps, an excellent place to show the value of logical equivalence.

The second book is mainly about calculus, though some geometry and trigonometry is included. The standard is higher than for Moore's book, and it would be desirable for the reader to possess a pretty good knowledge of the subject at the school level. Otherwise, when faced on p. 10 with the 16 axioms of the real number field, he may miss the motivation; further, he may wonder whether it is possible for a set of entities satisfying such restrictive conditions to exist at all! Again, unless he knows something of the power of the vector calculus, as far at least as the use of scalar and vector products, he may well wonder when the long and careful discussion of the definitions (20 pages) is going to do anything. However, if he is already familiar with the techniques, the careful examination of foundations will do him a great

deal of good. Basic ideas in the calculus, such as continuity and limits, are examined in great detail, and once the existence of a least upper bound is granted as an axiom, appeals to intuition are dismissed. Some novelties of notation are employed, possibly following Menger's ideas; it would be interesting to have these tried out by teachers in this country, and opinions obtained. These novelties mainly concern the clarification of the function concept; thus, I being the identity function $I(x) = x$, the authors write

$$\int I^2 \exp = (I^2 - 2I + 2) \exp;$$

and the rule for differentiation of a function of a function (the chain rule) appears in the form

$$D[f \cdot g] = [(Df) \cdot g]Dg.$$

The intention seems to be to revert to the standard notation in examples, while writing the theory in the new form; one would like to be sure that the clarification of ideas is not out-weighed by confusion in technique.

T. A. A. B.

School Geometry (Part 1); By F. E. Blamey; University Tutorial Press; 6/0 (with answers 6/6) pp. 195.

"The treatment is on the lines of the recommendations for "Stage A", set out in the reports of the Mathematical Association". This extract from the Preface sets the seal on the book. As an introduction to discovery by drawing and intuition, leading to the germ of logical deduction, this book is one of the best yet issued. There is a remarkable range and variety in the exercises; a pupil forced by necessity to start Geometry by himself could not possibly do better than work through as many of the examples in the book as he can find time for. He would obtain a first-class foundation even if he merely attempted a fraction of them. As a Class-book, the wealth of oral exercises will ensure that every pupil really grasps the point at issue. There are excellent chapters on symmetry, loci, and envelopes—the last containing some very interesting examples of paper-folding.

Only one small criticism can be made on a point of method. The "eye-brow" method of drawing parallels is not accepted by at least one G.C.E. Board as an accurate construction.

A school looking for a good introduction to Geometry should certainly consider this book. Secondary Modern Schools should find it particularly useful.

E. H. Copsey

Ordinary Level Algebra (Part 1); By T. H. Ward Hill; Harrap, 6/6 (with answers). pp. 144.

This book, the first of two parts, is by a very experienced author and teacher; his reputation is by no means diminished by it. The treatment follows the usual sequence—formulae, generalisation, manipulation of indices etc., brackets, equations, simple problems, fractions, directed numbers, long multiplication and division, finishing

with simultaneous equations. There are three sections on graphs, two arithmetical, and one dealing with the graph of a function.

The explanations are clear; even the multiplication of directed numbers is adequately dealt with. The exercises are numerous and well-arranged. There are revision examples at suitable intervals, and altogether the book is attractively written and printed. The chapter on long multiplication sets out the work in columns; the advantages of this method over that in which the working is done with brackets side by side has never been apparent to the reviewer. The second arrangement occupies less space, and older pupils never use the column method. It is felt that the column set-out could be discontinued altogether in the future. This criticism apart, the book is very good of its kind. E. H. COPSEY

Simplified calculus. By F. L. WESTWATER. Pp. xv, 160. 10s. 6d. 1960. (English Universities Press).

This breezy little book bears traces of the author's experience, as an Instructor Officer of the Royal Navy, of presenting calculus to those whose background of algebra and geometry is not very strong; a course has to be steered between an indigestible rigour and the sterile dogmatism of a set of working rules. Ideas which can be grasped intuitively and assisted by diagrams are accepted without formal proof, and space thus saved is given to a very full discussion of notation. The novice who reads the first three chapters thoroughly should appreciate that the somewhat bizarre standard notation is the result of experiment and convention, that it has survived because it works better than its obsolete competitors. The standard derivatives and integrals are dealt with and there are simple applications and a reasonable number of exercises; but there is a welcome insistence on the need to regard method as being as important as results. T. A. A. B.

The New World of Mathematics. By G. A. W. BOEHM, and the Editors of Fortune. Pp. 128. 10s. 6d. 1960. (Faber & Faber)

This is a book about mathematicians rather than about mathematics, written very much from the American standpoint. Apparently research in Mathematics is confined to the U.S.A., Moscow, Leningrad and Paris. Amongst the topics discussed are computing machines, linear programming, monte carlo methods, and very sketchily, topology, groups, dimension theory, matrices and the algebra of logic. The book does suceed in conveying something of the excitement of mathematical discovery. R. L. G.

Einführung in die Theorie der Spiele. By EWALD BURGER. Pp. 169. DM. 28. 1959. (Walter de Gruyter & Co., Berlin.)

This, the first German book on the Theory of Games, is remarkable for the organisation of its contents. Perhaps less inhibited by tradition than its counterparts written in English, it starts off with the general

concept of a game, listing oligopoly and other economic models, passes on to non-cooperative games and their equilibrium points, then specializes to two-person zero sum games (with about 20 pages on Linear Programming) and deals, finally, with the von Neumann-Morgenstern theory of cooperative games and their further developments, e.g. the Shapley value.

The exposition is mathematical and abstract, but introductory remarks explain well the rationality of the ideas and the rationale of the formal development. Pains are also taken to justify the remark on the title page: "with examples of applications, particularly from economics and sociology." Ample use has been made of papers in the collective volumes Annals of Mathematics Studies 24, 28, 38, 39 and 40.

Compared with what we expect in this country, the production is poor and makes the book difficult to read even for those conversant with the German language. The print is unattractive and formulae are not sufficiently well displayed. Also, the habit of denoting some theorems by numbers and others by letters does not help. The bibliography lists textbooks and collections by letters (e.g. NM for von Neumann-Morgenstern) and research papers by numbers. Only one of them, by Vogel on approximations to good strategies in Math. Zeitschr. *65* (1956) is not generally known in this country.

Libraries of Game Theory would not be complete without this up to date, mathematically thorough and well thought out treatise.

S. VAJDA

Axiomatic Set Theory. By P. BERNAYS and A. A. FRAENKEL. Pp. 226. 1959. (North Holland Publishing Co., Amsterdam.)

All who are interested in the foundations of set theory have been eagerly awaiting the publication of this book since it was first announced over four years ago. Apart from the brief historical introduction by Prof. Fraenkel, the book is entirely the work of Prof. Bernays. In the preface Bernays says that this system of set theory has been designed mainly for the use of mathematicians working in the field and not just as an object for metamathematical investigation. The system is a modification of one described in a series of famous papers in the Journal of Symbolic Logic between 1937 and 1954. A special feature of the present system is the employment of class variables only as free variables, which produces a stronger separation of classes from sets. Only the axiom of choice is stated in existential form, the remaining axioms being formulated in terms of function symbols, e.g. the potency axiom

$$c \in \pi(a) \leftrightarrow c \subseteq a$$

"the elements of $\pi(a)$ are the subsets of a" and the axiom of infinity

$$a \in \omega \leftrightarrow Nu(a)$$

"a belongs to the set ω of natural numbers if and only if a is an ordinal such that itself and every element of it is either 0 or a successor".

The theory of finite sets, and of natural numbers is based on the concept of ordinal, using a definition of ordinal due to R. M. Robinson. Various forms of the axioms of choice and of infinity are discussed, and amongst the definitions of finiteness the following is especially noteworthy: "A set is finite if and only if every ordering of it is a well ordering". There is a chapter on the construction of the theory of real numbers, using the principle of the least upper bound, and a study of the arithmetic of cardinal and ordinal numbers. Finally a one-one correspondence is established between the class of ordinals and the class of all sets.

The book is beautifully prepared and the author has spared no pains to make the readers' task as easy as possible; in addition to the general index, author index and bibliography there is a three page index of symbols.

In a second volume Professor Bernays proposes to discuss the relative consistency of the system here constructed, the independence of the axioms and the elimination of the descriptions operator.

R. L. GOODSTEIN

A Method in Proofs of Undefinability. By K. L. de BOUVÈRE. Pp. 64. 15s. 1959. (North Holland Publishing Co., Amsterdam.)

The problem discussed in this thesis is that of proving some operation in a formal system to be independent of the other operations of the system.

Recently interest has been focussed upon a method given by Padoa at the turn of the century for proving operational independence by finding two (consistent) interpretations of the formal system in both of which all operations but the operation in question have the same interpretation but this particular operation receives two different interpretations. de Bouvère contrasts this method with one in which a notion d in a formal system is proved independent of the other notions by finding a model of the formal system without d in which it can be proved that there is no way of introducing an interpretation of the notion d; this method is applied to show (amongst other results) that multiplication is not definable explicitly in terms of addition (in a formalisation of arithmetic), a result which is otherwise known indirectly from the facts that a formalised arithmetic with both addition and multiplication is not decidable, but a system with only one of these operations is decidable.

R. L. GOODSTEIN

Differential Geometry. By E. KREYSIG, Mathematical Expositions No. 11, University of Toronto Press, 1959. pp. xiv, 352, 48/-.

Except for the addition of a short bibliography, this is an English translation of the earlier book "Differentialgeometrie", Leipzig 1957, by the same author, which was reviewed in *Math. Gazette* 1959, p. **151**.

T. J. WILLMORE

Praxis der konformen Abbildung. By W. von KOPPENFELS and F. STALLMANN. Pp. xiii, 375. 1959 Grundlehren der mathematischen Wissenschaften, 100. (Springer)

Before dealing with this particular book, we must congratulate the great firm of Springer on the not-out century of their famous *Grundlehren* series. After nearly forty years, the delights of reading Blaschke and Hurwitz-Courant are still fresh; at about 10 shillings a copy, those volumes were not beyond the purse of a somewhat impecunious undergraduate. Throughout the series, a high standard of mathematical content and of printing craftsmanship has been fully maintained.

The present volume has its roots in the war years, when several belligerent countries realised the need for working manuals on conformal mapping. Koppenfels was one of a German team, and he produced a manuscript on the mapping of polygonal domains which forms the basis of the second part of the book now before us. Ullrich and Stallmann were commissioned by Springer in 1953 to look into the possibility of using this manuscript as the basis for a book on the subject, but Ullrich's death in 1957 threw the final responsibility on Stallmann.

The book is almost self-contained, for it begins with the definition of complex numbers and functions, with emphasis on the geometry of the subject; then the conformal mappings associated with the elementary functions are studied. The more general theory is then sketched; for full details of the proofs of the fundamental theorems, such as that of Riemann's, the reader is rightly referred to standard treatises. This occupies about 100 pages. The following 100 pages deal with polygonal mappings, including of course a full account of the Schwarz-Christoffel process, and ending with an interesting section on approximative methods applicable when the boundary is too complicated for exact transformations to be effective; some reference is made here to integral equation methods devised mainly in the U.S.A in connection with aerofoils. The last section of the book is a catalogue of mappings, arranged roughly in order of the number of straight or circular edges forming the boundary.

T. A. A. B.

Les Methodes en Génétique Générale et en Génétique Humaine. By R. HURON et J. RUFFIÉ. Pp. 556. 8200 francs. 1959. (Masson et Cie., Paris.)

The scope of this book is indicated by the title. A special feature is the detailed discussion of mathematical and statistical aspects, mostly on a fairly elementary level. Fundamental genetical and statistical ideas are all clearly defined. The book is up to date, and covers almost all statistical methods required in practice by the geneticist, including especially a good section on the detection of linkage. The mathematician who wishes to learn about genetics will find this an excellent introduction, but should be warned that a few parts are not universally accepted (e.g., "partial sex-linkage" in human genetics).

CEDRIC A. B. SMITH

Differenzengleichungen. By H. Meschkowski. Pp. 243. DM36. 1959. Vandenhoeck and Ruprecht, Göttingen.

This is volume 14 of the Studia Mathematica series of mathematical textbooks published under the auspices of the late Professor Süss's Forschungsinstitut at Oberwohlfach. The author devotes two chapters to the theory of difference equations with integral argument and the remainder of the book to problems of function theory and difference equations with any complex argument. The titles of the chapters are:— I Recursive sequences, II Power series methods, III Summation methods, IV Bernoulli polynomials, V Functions defined by difference equations, VI General theorems on linear difference equations, VII Linear equations with constant coefficients, VIII Poincaré's and Perron's theorems, IX The methods of Hurwitz and Ghermanesco, X Use of Laplace transforms, XI Integral representations of classical functions, XII Development of solutions in series, XIII An operator method for linear equations with rational coefficients, XIV Linear difference equations with general coefficients, XV Linear first order q-difference equations, XVI q-difference equations of general type, XVII Hölder's theorem, XVIII Interpolation problems. Each chapter is followed by a number of suitable examples. The exposition is clear and the book provides a very useful introduction to a subject of great and increasing importance. Chapters V and VIII are particularly valuable.

E. M. Wright

Elementare und Klassische Algebra, Vol. II. By Wolfgang Krull. Sammlung Göschen 933, 1959. (Berlin, Walter de Gruyter.)

In this and the preceding slim volume (same Sammlung 930) the author succeeds admirably in treating elementary algebra from an advanced, and classical algebra from a modern point of view. The first volume dealt mainly with elementary problems of the 16th, 17th and early 18th century and introduced the reader to the basic ideas of ring- and field-theory. The present volume gives an account of Galois theory and normal forms for non-metacyclic equations, algebraic topics of the 19th century. It gives an opportunity of becoming well acquainted with group- and vector space-theory. An overlap with the well-known textbooks in algebra is consciously avoided, as far as possible, and even a reader with a good knowledge of modern algebra will find a wealth of new and interesting material.

K. A. Hirsch

Matrix Calculus. By E. Bodewig. Second revised and enlarged edition. 1959. (North Holland Publishing Co., Amsterdam.)

This new edition which has become necessary after only three years, contains over 100 pages more than the first. New methods of dealing with the fundamental computational problems involving matrices are given and new applications in diverse fields.

K. A. Hirsch

Constructivity in Mathematics. Edited by A. HEYTING. Pp. 297. 60s. 1959. (North Holland Publishing Co., Amsterdam.)

This volume contains the lectures held at the International Colloquium in Amsterdam in 1957, together with summaries of some lectures given at Cornell University in the same year on related themes. The lectures represent a very wide range of views; papers by Heyting and Mostowski specifically discuss various degrees of constructivism.

Conspicuous unfortunately only by his absence was A. A. Markov, who was to speak on Russian work in recursive analysis.

R. L. GOODSTEIN

The structure and evolution of the universe. By G. J. WHITROW. Pp. 212. 21s. 1959. (Hutchinson)

This book is a thoroughly revised and appreciably extended version of Dr. Whitrow's *Structure of the Universe* (1949) reviewed in the *Gazette* (XXXIV, p. 148). The new edition is called for on the one hand because of the deserved popularity of the first edition and its translations into three other languages, and on the other hand because of the great amount of observational and theoretical work in the field during the past ten years. Dr. Whitrow has incorporated into the re-writing of his book a survey of the most significant features of recent work in a remarkably clear and concise fashion. As the modification of the title implies, the extension of the book has mainly to do with the evolution of stars and stellar systems on which only the latest researches have shed any fairly certain light. All the commendation that one felt bound to give to the earlier book can now be transferred to the new one. But the reader must be advised to read the work as a whole, for the book as well as its subject has a structure and the significance of each part can be properly appreciated only with reference to its place in the general development of the whole account.

W. H. MCCREA

BRIEF MENTION

Theory of Value. By G. DEBREU. Pp. 114. 38s. 1960 (Chapman & Hall)

An axiomatic Analysis of Economic Equilibrium, using topological concepts.

Statistical Independence in Probability, Analysis and Number Theory. By MARK KAC. Carus Mathematical Monographs No. 12. Pp. 93. 24s. 1959

Except for the last chapter which deals with an application of the ergodic theorem to continued fractions, the book is concerned to show how the notion of statistical independence arises naturally in various contexts cutting across different mathematical disciplines.

The Dynamics of Particles. By A. G. WEBSTER. Pp. 588. $2.35. 1959.

Advanced Calculus. By E. B. WILSON. Pp. 566. $2.35. 1959.

Symbolic Logic and the Game of Logic. By LEWIS CARROLL. Pp. 199 + 96. $1.50. 1959.

Pillow Problems and a Tangled Tale. By LEWIS CARROLL. Pp. 109 + 152. $1.50. 1959.

The Theory of Functionals and of Integral and Integro-Differential Equations. By V. VOLTERRA. Pp. 226. $1.75. 1959.

The Measurement of Power Spectra. By R. B. BLACKMAN and J. W. TUKEY. Pp. 190. $1.85. 1959. (Dover, New York. Constable, London.)

This very varied new collection of Dover reprints will bring many new readers to the series. We are justly reminded that Lewis Carroll earned his living as a mathematical Tutor at Oxford. The *Pillow Problems* are problems in elementary number theory and geometry; the *Tangled Tale* was originally published as a magazine serial, and the puzzles it propounds are provided with solutions. Lewis Carroll's *Symbolic Logic* presents graphical methods for solving problems of syllogistic logic, and the same methods are described in the form of a children's game in the *Game of Logic*.

Wilson's *Advanced Calculus* was first printed nearly 50 years ago and is a comprehensive text book on differential equations, multiple integrals and allied topics. Webster's *Dynamics* is justly celebrated for the clarity of its treatment of fundamental notions. Volterra's *Theory of Integral Equations* reproduces Whittaker's Obituary of Volterra, written for the Royal Society, and lists 270 of Volterra's publications. The *Measurement of Power Spectra* by Blackman and Tukey, is reprinted from the *Bell System Technical Journal*, with an Index and Preface prepared for this Edition.

A Course of Pure Mathematics. By G. H. HARDY. Students' Edition. Pp. 509. 22s.6d. 1959. (Cambridge University Press)

This paper-backed edition of a famous text book marks the Cambridge University Press's first venture into the field of paper-backed student texts. This volume is strongly made and opens flat at any page.

Conduction of Heat in Solids. By H. S. CARSLAW and J. C. JAEGER. 2nd Ed. Pp. 510. 84s. 1958. (Oxford University Press)

The 1946 Edition of this important work was reviewed in *Gazette* XXXVI, p. 142. In the second edition many new results have been added, including results on heat generation, surface heating, melting and freezing. Other additions include new tables and text-figures and two short survey chapters, on Fourier transforms and numerical methods.

Journal of Applied Mathematics and Mechanics. Vol. 22. No. 1. 1958. (Pergamon Press)

This translation of a Russian periodical signals the commencement of a new series of translations, intended to be carried out by Russian-speaking scientists familiar with the field. The American Society of Engineers are to be congratulated for initiating this valuable service.

Teachers' Information Guide and College of Preceptors' Calendar, 1958–59.

This guide contains a catalogue of Educational Magazines and Publishers and particulars of the College of Preceptors.

Linear Groups. By L. E. DICKSON. Pp. 312. $1.95. 1958. (Dover, New York. Constable, London)

In the introduction to this reprint of the original edition of 1901 Wilhelm Magnus says "L. E. Dickson's book on linear groups, with an exposition of the Galois Field Theory, is a milestone in the development of modern algebra .... it contains the most extensive and thorough presentation of Galois fields available in the literature. Although many of his proofs would be shortened today, and although the same subject would be expounded differently now, the first five chapters are still very readable and contain an enormous wealth of both examples and theorems."

Analytic Geometry of Three Dimensions. By G. SALMON. Pp. 470. $4.95. 1958

Finite Differences. By G. BOOLE. Pp. 336. $4.95. 1958

Combinatorik. By E. NETTO. Pp. 341. $4.95. 1958

Three new Chelsea reprints of considerable interest. The first two are so well known in this country that they need no recommendation. The third is a very valuable collection of results on binomial coefficients, with additional notes by Skolem and Brun of which the following formula due to Frisch may serve as an Example

$$\sum_{r=0} (-1)^r \binom{a}{r} \Big/ \binom{b+r}{c} = \frac{c}{a+c} \Big/ \binom{a+b}{b-c}$$

A Treatise in Plane and Advanced Trigonometry. By E. W. HOBSON. Pp. 383. $1.95. 1958

The Theory of Functions of a Real Variable. By E. W. HOBSON. Vol. I. Pp. 736. $3.00. Vol. II. Pp. 780. $3.00

These Dover reprints are remarkable value for the price. Hobson's trigonometry after nearly 70 years remains one of the best books of its kind. The Theory of Functions, out of date perhaps in many respects, is a great warehouse of information, in particular of the history of analysis.

The Fifth Mental Measurements Yearbook. Edited by O. K. BUROS. Pp. 1292. $22.50. 1959. (Gryphon Press)

Mathematical Tests are listed and reviewed on pp. 419–503, and include Tests by the National Foundation for Educational Research in England and Wales. Tests cover Arithmetic, Algebra and Trigonometry. Amongst the reviewers are M. F. Rosskopf, J. Wrigley, J. Sutherland and A. E. Meder.

Numerical Analysis and Partial Differential Equations. By G. E. FORSYTHE and P. C. ROSENBLOOM. Pp. 204. 60s. 1958. (Chapman and Hall)

This is two books bound up in one volume. G. E. Forsyth writes on the contemporary state of numerical analysis, emphasising the contributions made by Russian authors whose work is not fully known here. The book by P. C. Rosenbloom on partial differential equations deals primarily with advances in the field of linear partial differential equations since 1953 including the author's own contributions.

Mathematical Methods and Operations Research. By T. L. SAATY. Pp. 421. 77s.6d. 1959. (McGraw-Hill)

This is a very readable account of the parts of mathematics utilised in operations research, starting with an account of mathematical proof, and covering introductory sections on theory of games, probability and queuing theory, with a final essay on creativity. The book concludes with a large set of problems of many levels of difficulty.

The Teaching of Science in Post-Primary Schools. By E. J. SEARLE. Pp. 259. 30s. 1959. (Oxford University Press)

This is a report on work undertaken for the New Zealand Council for Educational Research. The author is a senior lecturer in Geology at Auckland University, and for 19 years was a Science Master at Auckland Grammar School. The report makes an assessment of what has been done in the past and offers suggestions for foundations for future development.

The History of the Calculus and its Conceptual Development. By C. B. BOYER. Pp. 346. 16s. 1959

The Theory of Numbers and Diophantine Analysis. By R. D. CARMICHAEL. Pp. 118. 11s. 1959

Mathematical Puzzles of Sam Lloyd. Edited by MARTIN GARDNER. Pp. 165. 8s. 1959

Theory of Flight. By R. VON MISES. Pp. 629. 22s. 6d. 1969

A Treatise of Algebraic Plane Curves. By J. L. COOLIDGE. Pp. 375. 21s. 1959

Fourier Series. By W. E. BYERLY. Pp. 287. 14s. 1959

Algebraic Theories. By L. E. DICKSON. Pp. 276. 12s. 1959. (Dover Publications, New York. Constable, London)

These Dover books are all unabridged and unaltered reprints. Boyer's book was first published under the title *The Concepts of the Calculus*. Carmichael's two books bound as one, treat such topics as congruences, primitive roots, rational triangles, and equations like $x^3 + y^3 = 2^m x^3$. This book is particularly well suited for school library use. Von Mises' *Theory of Flight* is an important introductory text. In this edition of Lloyd's puzzles the original illustrations have been reproduced, adding greatly to the books interest. Coolidge's Plane Curves has long been out of print and is a most welcome addition to the Dover series. Byerly's book treats not only Fourier Series but also spherical, cylindrical and ellipsoidal harmonics.

Differential Equations. By G. Boole. Pp. 735. $6.00. 1959. (Chelsea, New York)

This reprint of Boole's *Differential Equations* is made from the Fifth Edition, and contains a list of Boole's papers and books.

Symbolic Logic. By C. I. Lewis and C. H. Langford. Pp. 518. $2.00. 1960. (Dover, New York. Constable, London)

This reprint is a corrected and unabridged republication of the First Edition (1932) with an additional appendix by C. I. Lewis. This reprint is remarkable value for $2.00 and is particularly welcome both for its service as an introductory text and its continued interest to research workers.

Theory of Probability. By W. Burnside. Pp. 106. $1.00. 1960. (Dover, New York. Constable, London)

This is an unaltered reprint of the 1928 Edition which was published after Burnside's death, and contains A. R. Forsyth's obituary notice of Burnside written for the Royal Society.

Gewöhnliche Differential gleichungen. By J. Horn & H. Wittich. 6th Ed. Pp. 275. DM 14. 1960. (W. de Gruyter. Berlin)

This new Edition contains much additional material on differential equations of complex variables and the theory of Special Functions.

Mechanisation of Thought Processes. Vols. I, II, Pp. 980, 50s., for the two volumes. 1960. (Stationary Office)

Proceedings of a Symposium held at the National Physical Laboratory in November 1958. The Symposium covered such topics as artificial thinking, character and pattern recognition, mechanical language translation and automatic programming. A total of 32 papers was presented.

Proceedings of the International Congress of Mathematicians 1958. Edited by J. A. Todd. Pp. 573. 65s. 1960. (Cambridge University Press)

This volume contains the official record of the International Congress held in Edinburgh in August 1958, and the text of the one hour and half-hour addresses given by invitation of the Programme Committee. The short communications made by members at the Congress are not included but the names of the Speakers and the titles of their papers are listed.

Récréations Mathématiques. By E. Lucas. Pp. 254. 8 N.F. 1960. (Blanchard, Paris)

Problèmes. By C. G. Bachet. Pp. 242. 9 N.F. 1959. (Blanchard, Paris)

Two interesting reprints. Bachet is famous for his solution of the diophantine equation $ax - by = 1$, and for his edition of Diophantos which inspired Fermat.

BRANCH REPORTS

OXFORD AND DISTRICT BRANCH

Annual Report for 1959–1960

The Branch was formed in June 1959, and has had a very successful first season. The area served includes Reading, and we have also members from Newbury, Marlborough, Shrivenham, Cheltenham, Buckingham, Abingdon, Harwell, Henley, Wallingford, Brackley and their neighbourhoods. The total membership is 161. Of the Senior Members 20 are from the Universities and 87 from the Schools. There are 54 Junior Members. The average attendance at meetings has been about 90.

Monday, 8th June, 1959. The Inaugural Meeting of the Branch was held. The following were elected to the Committee:—

President: Professor C. A. COULSON.

Vice-President: Mr. D. A. QUADLING.

Secretary: Mr. F. A. GARSIDE. *Treasurer:* Mr. A. R. TAMMADGE.

Committee, Dr. J. M. HAMMERSLEY (representing Oxford University), Miss E. J. TERNOUTH (representing Reading University), Mr. J. S. BUTLER, Mr. N. R. EYRES, Dr. J. HOWLETT, Miss J. M. MATHEWS, and Mr. I. M. HALL (Student representative).

Thursday, 18th June, 1959. Professor C. A. COULSON gave his Presidential Address on the subject 'Likes and Dislikes in Mathematics'.

Wednesday, 24th June, 1959. Visit to the Hydraulics Research Establishment, Wallingford.

Friday, 23rd October, 1959. Professsor R. RADO addressed the Branch on 'Mathematics: the art of proving the improbable and disproving the obvious'.

Wednesday, 25th November, 1959. The Branch met at Reading University. Dr. J. M. HAMMERSLEY gave a lecture on 'The shape of Mathematics to come?'

Friday, 22nd January, 1960. Dr. W. L. FERRAR addressed the Branch on 'The difficulties of writing Mathematical text-books'.

Tuesday, 1st March, 1960. The Branch met at Radley College. Mr. G. J. R. POTTER gave a short address on 'The Administrative side of Examinations'. This was followed by a lengthy and most useful discussion. This meeting was not open to Junior Members.

Thursday, 5th May, 1960. Dr. H. MARTYN CUNDY gave a lecture on 'The Mathematics of And/Or'.

Thursday, 2nd June 1960. The Annual General Meeting was held. The following were elected to serve on the Committee for 1960–1961:—

President: Professor R. RADO.

Vice-President: Mr. D. A. QUADLING.

Secretary: Mr. F. A. GARSIDE. *Treasurer:* Mr. A. R. TAMMADGE.

Committee: Dr. J. M. HAMMERSLEY (Oxford University), Dr. L. A. G. DRESEL (Reading University) Mr. J. S. BUTLER, Mr. N. R. EYRES, Dr. E. A. WOODALL, Mr. A. M. WOOD, and Mr. P. M. NEUMANN (Student representative).

During the year the Oxford University Institute of Education one day Conference for Mathematics specialists teaching in Grammar and

Public Schools was arranged in cooperation with the Branch, and was attended by many members (23rd November, 1959). Experiments were also made by the Branch in organising special lectures for Sixth Form Mathematicians and Scientists. These took place at Newbury (Lecturer Professor C. A. Coulson), Chipping Norton (Dr. J. Howlett and Mr. D. G. Kendall), Marlborough (Dr. A. B. Tayler and Oxford (Mr. D. G. Kendall). The experiment appears to have had great success, and we hope to pursue the idea in the light of the experience gained.

F. A. GARSIDE, Hon. Sec.

LEICESTER AND COUNTY BRANCH

Report for the Session 1959–1960

At the first meeting of the session Mr. A. P. ROLLETT, H.M.I. was to have opened a discussion on the Ministry of Education's pamphlet on the Teaching of Mathematics. Unfortunately Mr. Rollett was prevented by illness from coming to Leicester and the discussion was opened by four members of the branch each commenting on one chapter of the report.

It was illness, again, that prevented Mr. T. R. GODDARD of the Leicester Education Authority from speaking to the Branch in November on "Mathematics in the Junior School". His place was taken at very short notice by Mr. L. G. W. SEALEY of the County Authority.

In January Miss L. D. ADAMS opened a discussion on the recently published Association Report on "Mathematics in the Secondary Modern School". All interested teachers, whether branch members or not, were invited to this meeting.

At other meetings speakers and subjects were as follows:

Mr. E. C. WITCOMBE, a former member of the branch, now headmaster of Wednesbury High School,—"A Mathematical Miscellany"—a talk addressed primarily to members of sixth forms;

Dr. W. L. FERRAR, Principal of Hertford College, Oxford, on "Difficulties in Teaching Algebra"; and

Mr. J. M. HAMMERSLEY, of Trinity College Oxford and the Atomic Energy Research Establishment, Harwell, on "An Elementary Introduction to Industrial Quality Control".

The Mathematical Quiz between teams from the sixth forms of Grammar Schools in the City and County was held as usual in November. This is now firmly established as an annual event and this year attracted an even larger audience than usual.

The Annual General Meeting in February was followed by short papers given by three branch members. At this meeting the following office holders were elected for the 1960–1961 session:

President: Dr. E. J. F. PRIMROSE.

Vice-Presidents: Professor R. L. GOODSTEIN, Mr. W. FLEMMING, Mr. J. W. HESSELGREAVES. *Treasurer:* Mr. L. G. CLARKE.

W. E. REMINGTON. *Hon. Sec.*

NORTH-EASTERN BRANCH

Report for the Session 1959–60

The following meetings were held:—

1st October, 1959

Mr. W. L. JAMES, Headmaster of Bedlington Grammar School, spoke on "The Teaching of Statistics to Sixth Forms".

9th November, 1959

Miss L. D. ADAMS, President of the Association, addressed branch members and guests on "The Teaching of Mathematics in Secondary Modern Schools".

5th December, 1959

Dr. E. A. FREEMAN, of Sunderland Technical College, gave a talk and demonstration on "Automatic Control Systems".

21st January, 1960

Mr. T. MCGREGOR, President of the Branch, gave an address entitled "Some Remarks on Teaching Mathematics in Secondary Schools".

19th March, 1960

This meeting was held in conjunction with the University of Durham School Examinations Board Annual Conference.

In the morning Professor L. ROSENHEAD, of Liverpool University, spoke on "Mathematics and Mathematicians in Science and Industry". In the afternoon there was a discussion on Advanced and Scholarship Level examination papers and syllabuses.

In addition to the normal branch meetings, the following open lectures were arranged by the University of Durham Extra-Mural Department at the suggestion of the branch.

4th February, 1959

Professor W. W. ROGOSINSKI, of King's College, Newcastle-upon-Tyne, spoke on "Infinity in Mathematics". This lecture was extremely successful, and was repeated on 11th February, 1959 and again on 18th February, 1959. The total attendance, mainly of sixth form pupils, exceeded 400.

9th March, 1960

Dr. E. S. PAGE, Director of the University of Durham Computing Laboratory, spoke on "Automatic Computors". This lecture was repeated on 23rd March, 1960.

J. F. REED, Hon. Sec.

QUEENSLAND BRANCH

Report for the Year 1959–60

At the Annual Meeting held on 23rd March, 1959, the following officers were elected for 1959–60:

President: Professor C. S. DAVIS.

Vice-Presidents: Messrs. H. M. FINUCAN, A. W. YOUNG.

Hon. Secretary-Treasurer: Mr. M. P. O'DONNELL.

Members of Committee: Miss M. A. MACLEAN, Messrs. I. A. EVANS, G. L. HUBBARD, Associate Professor J. P. MCCARTHY, Mr. P. B. MCGOVERN.

The Presidential Address, entitled "The Natural Numbers", was given by the retiring President, Professor C. S. DAVIS.

Three general meetings were held during the year. On 14th May, Mr. I. A. EVANS read a paper on "Arithmetic in Music". On 27th July, papers on "The Teaching of Senior Algebra" were read by Messrs. G. T. EVANS, H. M. FINUCAN and R. MACKIE. On 6th October, Professor M. H. STONE, of the University of Chicago, read a paper on "American Approaches to some Problems of Mathematical Instruction".

The membership of the branch is at present 60, of whom 24 are members of the Mathematical Association. The average attendance at meetings during 1959 was 35.

M. P. O'DONNELL, Hon. Sec.

THE MATHEMATICAL ASSOCIATION

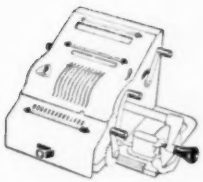
The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

Change of address should be notified to the Membership Secretary, Mr. R. E. Green. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

Subscriptions should be paid to the Hon. Treasurer of the Mathematical Association.

The **Library** of the Mathematical Association is housed in the University Library, Leicester.

The address of the Association and of the Hon. Treasurer and Secretaries is **Gordon House, 29 Gordon Square, London, W.C.1.**



Dear Member,

AN INVITATION

At the last Annual Meeting of the Association I was privileged to exhibit the range of Brunsviga hand and electric calculating machines particularly suitable for teaching purposes.

I believe this is the first time that calculating machines have been shown at the Annual Meeting and I would like to thank the Honorary Officers for making this possible.

During my ten years membership of the Association I have seen it grow rapidly and this is just another instance of the forward-looking policy of the Association in recognizing the importance of calculating machines in teaching mathematics, whether it be to the mathematical specialist or to science and engineering students.

The numerous enquiries I received at the meeting have encouraged me to make this invitation to Members of the Association:

A hand Brunsviga calculating machine will be sent on trial for 14 days on request on your letter heading.

I shall be very pleased to hear from you.

Yours very truly,

M. E. SHOENBERG

Educational Advisor,
Brunsviga Division,
Olympia Business Machines Co. Ltd.,
35 Red Lion Square,
London W.C.1. (*Phone:* HOLborn 2274)

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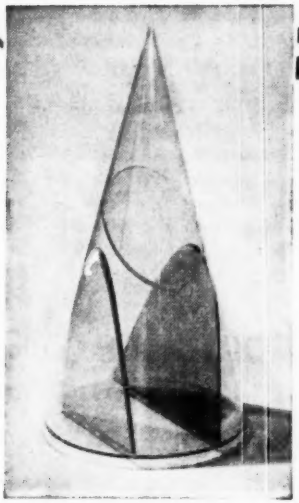

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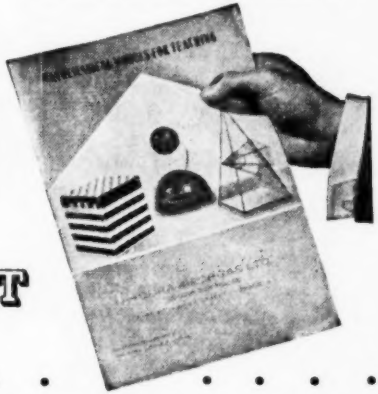

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OXFORD UNIVERSITY PRESS

EUREKA

THE JOURNAL OF THE ARCHIMEDEANS

Eureka is published annually in October by the mathematical students at Cambridge University. Its articles and puzzles are not all in serious vein and may appeal to many sixthformers as well as to their seniors. In the latest issue, No. 23, you may also read of further adventures of Gulliver, this time in Applyland, by Prof. Karl Menger, and all about pentominoes by Dr. J. C. P. Miller.

Postal subscribers may either pay for each issue on receipt or, by advancing 10s. or more, receive future issues as published at approximately 25% discount, with notification when credit has expired. The current rates are: payment on receipt 2s. 6d. (+ 2d. postage), in advance 2s. 0d. post free. Cheques and postal orders should be made payable to "The Business Manager, *Eureka*" and sent to The Arts School, Bene't Street, Cambridge.

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THE DIPLOMA IN MATHEMATICS

The first examination for this diploma, to be awarded annually by the Mathematical Association, will begin on November 13, 1961, and will be held simultaneously at a number of Local Centres in the areas served by the Branches of the Association. Candidates wishing to enter for the Examination are required to register with the Examination Secretary, The Mathematical Association, 29 Gordon Square, London, W.C.1, not later than March 13, 1961. Regulations concerning the Diploma are available from the secretary at the above address, from whom also specimen papers will be obtainable shortly at a cost of 2s. per set.

SUBSCRIPTIONS AND INCOME-TAX

The Hon. Treasurer of the Mathematical Association wishes to draw the attention of members to the fact that Income Tax is now recoverable by members on the amount of their Annual Subscription if they make a personal claim with their local Tax authority. The Covenant scheme is, however, still open to members by which they may sign an agreement to pay the full subscription to the Association for seven years and forgo their personal claim for refund of income tax. The Association can then recover the full Income Tax on such sum as will, after the deduction of tax, leave the full net subscription to be paid by the member. Put simply, if the member forgoes his tax rebate of about 8s., and signs a seven-year covenant, the M.A. gains over 13s. annually at present rates of tax.

MATHEMATICAL ASSOCIATION

REPORT OF THE COUNCIL FOR THE YEAR 1959

MEMBERSHIP

During the year ended 31 October 1959, 338 ordinary members and 195 junior members joined the Association, and at the end of the year membership figures were: Honorary 9; Ordinary 3265; Junior, 318; Life, 261. Total membership was thus 3853, showing an increase of 533 during the year.

The Council regrets to report the death of the following members: Mr. W. C. Fletcher, C.B. (1910), (President 1939–43); Mr. P. Fraser (1912), Mr. S. T. R. Hancock (1942); Mr. J. A. Johnston (1908), Mr. W. S. Lake (1947), Rev. Br. D. F. O'Connell (1944), Miss F. A. Preston (1946), Mr. D. W. Simmons (1947) and Mr. H. P. Sparling (1913).

The Council regrets also the death, since the year under review, of Mr. A. W. Siddons (1899), President 1935.

FINANCE

The Treasurer's statement for the year ended 31 October, 1959, shows an excess of Income over Expenditure of £244.12.2d.

The total Income of £8,377.12.10d includes the second and final grant from the Gulbenkian Trust and £1,352.19.8d received from the Commissioners of Inland Revenue. This latter sum represents the payments made under covenanted subscriptions for the years 1956–59 following the success of the appeal in April to the Special Commissioners.

The total Expenditure during the year amounted to £8,133.0.8d the highest in the history of the Association. The most important items are £2,872.5.5d spent on printing the Secondary Modern Schools' Report, £1,336.3.9d in reprinting seven other Reports and £2,977.15.3d in printing and distributing the Gazette. In addition, the Association has been able to repay the whole of the loans from members, amounting in all to £1,434.17.9d.

The financial position can now be regarded as sound, following the difficulties of the past two or three years, and tribute should be paid to Mr. J. B. Morgan, Treasurer for the past ten years, whose wise guidance has made the present satisfactory situation possible.

THE MATHEMATICAL GAZETTE

The distribution of the Gazette has grown so rapidly that increased printing has several times failed to keep up with

increased demand. For the first time, the printing of an issue has reached the 5,000 mark. There has also been a valuable increase in demand for advertisement space. The new section on "Class Room Notes" continues to be well supported, but "Beginners' Corner" has not yet received the support necessary to secure its continued existence. A problem section is beginning to develop of its own accord.

Library

The Library has had a fairly quiet year with few accessions apart from periodicals. The value of the Library for research in Educational methods in Mathematics is perhaps not fully appreciated in this Country, but a research student from Norway, who spent a fortnight working in the library during the Summer, found the collection of great value.

A selection of the Association's books loaned for an Exhibition held at a Technical College in connection with a Summer Course for Mathematics Teachers.

Teaching Committee

The work of the various sub-committees has continued normally. A strong sub-committee has been considering the revision of the Arithmetic Report, and it is not unlikely that a Second Arithmetic Report may appear.

Cloth-bound copies of the Modern Schools Report became available for public sale in July. A further reprint of the Primary Report will be available in January, and a reprint of the Technical Colleges Report is being considered.

The Branches

Two new Branches have been formed during the year. One, meeting in Oxford and drawing its support from both the Oxford and Reading areas, has started with an impressively large membership and an ambitious programme of meetings. A second new Branch has been formed in New Zealand and we await reports of its activities with interest.

Reports from well-established Branches all indicate full programmes and well-maintained membership rolls. The practice of interchanging speakers as a result of information circulated round the Branches has helped to maintain the high standard of branch meetings and has eased the work of programme committees.

Problem Bureau

The Bureau has been very active in the last few months, after a period of average activity. It has been found necessary

to impose a limit of *six* to the number of solutions which will be sent at any one time. Many interesting problems have appeared in recent scholarship papers, and there have also been questions asking for the candidate's ideas on a general topic. These are not "problems". The "staff" has shown its efficiency and tenacity by providing a number of elegant solutions to extremely awkward problems.

OFFICERS and COUNCIL

Council wishes to record its thanks to the President, Miss L. D. Adams, and to the Officers, for their efforts on behalf of members. The task of conducting the affairs of the Association is no light one; the year has seen greatly enhanced activities, and some consideration is to be given to a possible reorganization of the work of the Officers in order to spread the burden. Particular mention must be made of the services given by Mr. M. A. Porter, membership secretary for several years.

An Introduction to Homological Algebra

D. G. NORTHCOTT

Homological algebra is relevant to many branches of pure mathematics, including number theory, geometry, group theory and ring theory. Professor Northcott's book will be useful to specialists as a reference work, and it can also be used as a textbook for an advanced course.

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